Studia Scientiarum Mathematicarum Hungarica 41 (3), 309–324 (2004)

A REMARK ON A THEOREM OF THE GOLDBACH–WARING TYPE

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Communicated by I. Z. Ruzsa

Abstract

Let p_i, $2 \leqq i \leqq 5$ be prime numbers. It is proved that all but $\ll x^{23027/23040+\varepsilon}$ even integers $N \leqq x$ can be written as $N = p_1^2 + p_2^3 + p_3^4 + p_5^4$.

1. Introduction and statement of results

In the thirties, I. M. Vinogradov [10] and Hua [5] established many fundamental theorems in additive prime number theory. Their methods were consecutively applied to various problems in additive number theory. Among others, Prachar established in 1952, [8] the following result:

There exists a constant $c > 0$ such that all but $\ll x(\log x)^{-c}$ even integers N smaller than x are representable as

$$N = p_1^2 + p_2^3 + p_3^4 + p_5^4 \tag{1.1}$$

for prime numbers p_i.

In [1] and [2], this theorem was improved as follows:

All but $\ll x^{19193/19200+\varepsilon}$ positive even integers smaller than x can be represented as in (1.1).

Here we improve upon this result by showing the following theorem:

THEOREM. *All but $\ll x^{23027/23040+\varepsilon}$ positive even integers smaller than x can be represented as in* (1.1).

2. Notation and structure of the proof

We will choose our notation similar as in [2]. By k we will always denote an integer $k \in \{2, 3, 4, 5\}$, by p we denote a prime number and L denotes $\log x$. c is an effective positive constant and ε will denote an arbitrarily

2000 *Mathematics Subject Classification.* Primary 11P32, 11L07.

Key words and phrases. Goldbach–Waring type theorems.

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small positive number; both of them may take different values at different occasions. For example, we may write

$$L^c L^c \ll L^c, \quad x^\varepsilon L^c \ll x^\varepsilon.$$

$d(n)$ denotes the number of divisors of n and $[a_1, \dots, a_n]$ denotes the least common multiple of the integers $a_1, \dots, a_n$. Be further

$$r \sim R \Leftrightarrow R/2 < r \leqq R, \quad \sum_{\chi \bmod q}^{*} = \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}}, \quad \sum_{1 \leqq a \leqq q}^{*} = \sum_{\substack{1 \leqq a \leqq q \\ (a,q)=1}}^{q}.$$

We set

$$P = N^{\frac{13}{180} - \varepsilon}, \quad Q = NP^{-1}L^{-E} \quad (E > 0 \text{ will be defined later}),$$

and

$$\mu = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - 1.$$

We define for any characters χ, χ_j (mod q), $q \leqq P$ and a fixed integer N:

$$C_k(a, \chi) = \sum_{l=1}^{q} \chi(l) e\left(\frac{al^k}{q}\right), \qquad C_k(a, \chi_0) = C_k(a, q).$$

$$Z(q, \chi_2, \chi_3, \chi_4, \chi_5) = \sum_{h=1}^{q}{}^{*} e\left(\frac{-hN}{q}\right) \prod_{k=2}^{5} C_k(h, \chi_k),$$

$$Y(q) = Z(q, \chi_0, \chi_0, \chi_0, \chi_0), \quad A(q) = \frac{Y(q)}{\phi^4(q)}.$$

When the variable N is fixed, we will always write $A(q)$ and neglect the dependency of $A(q)$ on N. Otherwise, we will write $A(q, n)$.

$$s(p) = 1 + \sum_{\alpha \geqq 1} A(p^\alpha), \quad S_k(\lambda) = \sum_{\sqrt[k]{x}/2^{k+1} < n \leqq \sqrt[k]{x}} \Lambda(n) e(n^k \lambda),$$

$$S_k(\lambda, \chi) = \sum_{\sqrt[k]{x}/2^{k+1} \leqq n \leqq \sqrt[k]{x}} \Lambda(n) \chi(n) e(n^k \lambda), \quad T_k(\lambda) = \sum_{\sqrt[k]{x}/2^{k+1} \leqq n \leqq \sqrt[k]{x}} e(n^k \lambda),$$

$$W_k(\lambda, \chi) = S_k(\lambda, \chi) - E_0 T_k(\lambda, \chi), \quad E_0 = \begin{cases} 1, & \text{if } \chi = \chi_0, \\ 0, & \text{otherwise.} \end{cases}$$

Using the circle method we define the major arcs M and minor arcs m as follows:

$$M = \sum_{q \leqq P} \sum_{a=1}^{q}{}^{*} I(a,q), \quad I(a,q) = \left[\frac{a}{q} - \frac{1}{Qq}, \frac{a}{q} + \frac{1}{Qq}\right],$$

$$m = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus M.$$

Let

$$R(N) = \sum_{\substack{\sqrt[k]{x}/2^{k+1} \leqq n_k \leqq \sqrt[k]{x},\ k \in \{2,\dots,5\} \\ n_2^2 + \cdots + n_5^5 = N}} \Lambda(n_2) \dots \Lambda(n_5).$$

Then we find

$$R(N) = \int_{\frac{1}{Q}}^{1+\frac{1}{Q}} e(-N\alpha) \prod_{k=2}^{5} S_k(\alpha)\, d\alpha \tag{2.1}$$

$$= \left(\int_M + \int_m\right) e(-N\alpha) \prod_{k=2}^{5} S_k(\alpha)\, d\alpha =: R_1(N) + R_2(N).$$

Arguing as in [2], we see that

$$I_2(N) \ll N^{\mu} L^{-A} \tag{2.2}$$

for any $A > 0$ and all but $\ll x^{1+2\varepsilon} P^{-1/128} < x^{23027/23040+3\varepsilon}$ even integers $x/2 \leqq N < x$. In the sections 3 and 4 we will show that for any given $A > 0$

$$R_1(N) = \frac{1}{120} P_0 \prod_{p \leqq P} s(p) + O\left(x^{\mu} L^{-A}\right), \tag{2.3}$$

where

$$x^{\mu} \ll P_0 := \sum_{\substack{m_1+m_2+m_3+m_4=N \\ x/2^{k(k+1)} < m_k \leqq x}} \prod_{k=2}^{5} \frac{1}{m_k^{1-\frac{1}{k}}} \ll x^{\mu} \quad \text{for} \quad N \in (x/2, x]. \tag{2.4}$$

Using that

$$\prod_{p \leqq P} s(p) \gg (\log P)^{-960},$$

(see p. lemma 4.5 in [1]), the theorem follows from (2.1)–(2.4).

3. The major arcs

We will make use of the following lemmas:

LEMMA 3.1. *Let $f(x)$, $g(x)$ and $f'(x)$ be three real differentiable and monotonic functions in the interval $[a,b]$. If $|f'(x)| \leqq \theta < 1$, $g(x)$, $g'(x) \ll 1$, then*

$$\sum_{a<n\leqq b} g(n)e\big(f(n)\big) = \int_a^b g(x)e\big(f(x)\big)\,dx + O\left(\frac{1}{1-\theta}\right).$$

PROOF. See lemma 4.8 in [9]. □

LEMMA 3.2. *For primitive characters $\chi_i \bmod r_i$, $(i = 1,2,3,4)$ and the principal character $\chi_0 \bmod q$ we have*

$$\sum_{\substack{q\leqq P\\ r|q}} \frac{|Z(q,\chi_0\chi_1,\chi_0\chi_2,\chi_0\chi_3,\chi_0\chi_4)|}{\phi^4(q)} \ll r^{-1+\varepsilon}(\log P)^c,$$

where $r = [r_1, r_2, r_3, r_4]$.

PROOF. This is lemma 3.3 in [2]. □

LEMMA 3.3.

$$\sum_{q>x} |A(n,q)| \ll x^{-1+\varepsilon} d(n).$$

PROOF. The proof follows literally the proof of lemma of (4.12) in [6]. □

LEMMA 3.4. *For $P \leqq x^{13/80-\varepsilon}$ there is*

$$\sum_{N\leqq x} \left| \prod_{p\leqq P} s(p,N) - \sum_{q\leqq P} A(q,N) \right| \ll xP^{-1/3+\varepsilon}, \tag{3.1}$$

which implies that

$$\prod_{p \leqq P} s(p, N) = \sum_{q \leqq P} A(q, n) + 0(x^{-\varepsilon}) \tag{3.2}$$

for all but $\ll x^{1+2\varepsilon} P^{-1/3}$ *even integers* N *with* $1 \leqq N \leqq x$.

PROOF. This theorem is stated in [2] for all $P \leqq x^{7/150-\varepsilon}$. The proof shows however that it holds for $P \leqq x^{13/80-\varepsilon}$ as well. Splitting the summation over n in residue classes modulo q, we obtain

$$S_k\left(\frac{a}{q} + \lambda\right) = \frac{C_k(a,q)}{\phi(q)} T_k(\lambda) + \frac{1}{\phi(q)} \sum_{\chi \bmod q} C_k(a,\chi) W_k(\lambda,\chi) + O(L^2).$$

Thus we obtain from (2.1)

$$R_1(N) = R_1^m(N) + R_1^e(N) + O(x^{\mu} L^{-A}) \quad (\text{for any } G > 0), \tag{3.3}$$

where

$$R_1^m(N) = \sum_{q \leqq P} \frac{1}{\phi^4(q)} \sideset{}{^*}\sum_{1 \leqq a \leqq q} \int_{-1/Qq}^{1/Qq} \prod_{k=2}^{5} C_k(a,q) T_k(\lambda) e\left(-\frac{a}{q}N - \lambda N\right) d\lambda,$$

$$\begin{aligned} &R_1^e(N) \\ &= \sum_{k=2}^{5} \sum_{q \leqq P} \frac{1}{\phi^4(q)} \sideset{}{^*}\sum_{1 \leqq a \leqq q} \int_{-1/Qq}^{1/Qq} \prod_{\substack{l=2 \\ l \neq k}}^{5} C_l(a,q) T_l(\lambda) \\ &\quad \times \sum_{\chi \bmod q} C_k(a,q) W_k(\lambda,\chi) e\left(-\frac{a}{q}N - \lambda N\right) d\lambda \\ &+ \sum_{\substack{k,l=2 \\ k<l}}^{5} \sum_{q \leqq P} \frac{1}{\phi^4(q)} \sideset{}{^*}\sum_{1 \leqq a \leqq q} \int_{-1/Qq}^{1/Qq} \prod_{m \in \{k,l\}} C_m(a,q) T_m(\lambda) \\ &\times \prod_{\substack{o=2 \\ o \neq k \\ o \neq l}}^{5} \sum_{\chi \bmod q} C_o(a,\chi) W_o(\lambda,\chi) e\left(-\frac{a}{q}N - \lambda N\right) d\lambda \end{aligned}$$

$$+\sum_{k=2}^{5}\sum_{q\leqq P}\frac{1}{\phi^4(q)}\sideset{}{^*}\sum_{1\leqq a\leqq q}\int\limits_{-1/Qq}^{1/Qq}C_k(a,q)T_k(\lambda)$$

$$\times\prod_{\substack{l=2\\ l\neq k}}^{5}\sum_{\chi \bmod q}C_l(a,q)W_l(\lambda,\chi)e\left(-\frac{a}{q}N-\lambda N\right)d\lambda$$

$$+\sum_{q\leqq P}\frac{1}{\phi^4(q)}\sideset{}{^*}\sum_{1\leqq a\leqq q}\int\limits_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}\sum_{\chi \bmod q}C_k(a,\chi)W_k(\chi,\lambda)e\left(-\frac{a}{q}N-\lambda N\right)d\lambda,$$

$$=: S_1+S_2+S_3+S_4.$$

We first calculate $R_1^m(N)$. Applying lemma 3.1 yields

$$T_k(\lambda)=\int\limits_{\sqrt[k]{x}/2^{k+1}}^{\sqrt[k]{x}}e(\lambda u^k)\,du+O(1)=\frac{1}{k}\int\limits_{x/2^{k+1}}^{x}v^{\frac{1}{k}-1}e(\lambda v)\,dv+O(1)$$

$$=\frac{1}{k}\sum_{x/2^{k+1}<m\leqq x}\frac{e(\lambda m)}{m^{1-\frac{1}{k}}}+O(1).$$

Substituting this in $R_1^m(N)$ we see

$$R_1^m(N)=\frac{1}{120}\sum_{q\leqq P}A(q)\int\limits_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}\left(\sum_{x/2^{k+1}<m\leqq x}\frac{e(\lambda m)}{m^{1-\frac{1}{k}}}\right)e(-N\lambda)\,d\lambda$$

$$+O\left(\sum_{q\leqq P}|A(q)|\prod_{k=2}^{5}\int\limits_{1/Qq}^{-1/Qq}\max\left(\left|\sum_{x/2^{k+1}<m\leqq x}\frac{e(\lambda m)}{m^{1-\frac{1}{k}}}\right|,1\right)d\lambda\right).$$

Using lemma 3.3 and the trivial bound

$$\sum_{x/2^{k+1}<m\leqq x}\frac{e(\lambda m)}{m^{1-\frac{1}{k}}}\ll\min\left(\sqrt[k]{x},\frac{1}{x^{1-\frac{1}{k}}|\lambda|}\right), \tag{3.4}$$

we derive using lemma 3.4,

$$(3.5)\quad R_1^m(N) = \frac{1}{120}\sum_{q\leqq P} A(q)\int_{-1/2}^{1/2}\prod_{k=2}^{5}\Bigg(\sum_{x/2^{k+1}<m\leqq x}\frac{e(\lambda m)}{m^{1-\frac{1}{k}}}\Bigg)e(-N\lambda)\,d\lambda$$

$$+ O\Bigg(\sum_{q\leqq P}|A(q)|\int_{1/Qq}^{1/2}\frac{1}{x^{3-\mu}|\lambda|^4}\,d\lambda\Bigg) + O(x^{\mu}L^{-A})$$

$$= \frac{1}{120}P_0\sum_{q\leqq P}A(q) + O\big((PQ)^3x^{\mu-3}L^c\big) + O(x^{\mu}L^{-A})$$

$$= \frac{1}{120}P_0\prod_{1\leqq p\leqq P}s(p) + O(x^{\mu}L^{-A}),$$

for all but $x^{1+2\varepsilon}P^{-1/3}$ integers $N\leqq x$, where P_0 is defined as in (2.4) and E is chosen sufficiently large in $Q = NP^{-1}L^{-E}$. In the sequel $E = E(G)$ is fixed. Now we estimate the terms S_i, $i = 1,2,3,4$. Using lemma 3.2 we can estimate S_4 in the following way:

$$|S_4|$$

$$= \Bigg|\sum_{q\leqq P}\frac{1}{\phi^4(q)}\sum_{\chi_2 \bmod q}\ \sum_{\chi_3 \bmod q}\ \sum_{\chi_4 \bmod q}\ \sum_{\chi_5 \bmod q} Z(q,\chi_2,\chi_3,\chi_4,\chi_5)$$

$$\times \int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}W_k(\lambda,\chi_j)e(-N\lambda)\,d\lambda\Bigg|$$

$$\leqq \sum_{r_2\leqq P}\ \sum_{r_3\leqq P}\ \sum_{r_4\leqq P}\ \sum_{r_5\leqq P[r_2,r_3,r_4,r_5]\leqq P}\ \sideset{}{^*}\sum_{\chi_2 \bmod r_3}\ \sideset{}{^*}\sum_{\chi_3 \bmod r_3}\ \sideset{}{^*}\sum_{\chi_4 \bmod r_4}\ \sideset{}{^*}\sum_{\chi_5 \bmod r_5}$$

$$\times \int_{-1/Q[r_2,r_3,r_4,r_5]}^{1/Q[r_2,r_3,r_4,r_5]}\prod_{k=2}^{5}|W_k(\lambda,\chi_k)|\,d\lambda \sum_{\substack{q\leqq P\\ [r_2,r_3,r_4,r_5]|q}}\frac{|Z(q,\chi_2\chi_0,\chi_3\chi_0,\chi_4\chi_0,\chi_5\chi_0)|}{\phi^4(q)},$$

$$\ll L^c \sum_{r_2 \leqq P} \sum_{r_3 \leqq P} \sum_{r_4 \leqq P} \sum_{r_5 \leqq P} [r_2, r_3, r_4, r_5]^{-1+\varepsilon}$$

$$\times \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \sideset{}{^*}\sum_{\chi_4 \bmod r_4} \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \int\limits_{-1/Q[r_2,r_3,r_4,r_5]}^{1/Q[r_2,r_3,r_4,r_5]} \prod_{k=2}^{5} \left| W_k(\lambda, \chi_k) \right| d\lambda.$$

Using $[r_2, r_3, r_4, r_5] \geqq r_2^{6\varepsilon} r_3^{1/13-2\varepsilon} r_4^{4/13-2\varepsilon} r_5^{8/13-2\varepsilon}$, we obtain

$$(3.6) \qquad S_4$$

$$\ll L^c \sum_{r_2 \leqq P} r_2^{-\varepsilon} \sideset{}{^*}\sum_{\chi_k \bmod r_k} \max_{|\lambda| \leqq 1/r_2 Q} \left| W_2(\lambda, \chi_2 \right|$$

$$\times \sum_{r_3 \leqq P} r_3^{-1/13+2\varepsilon} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \max_{|\lambda| \leqq 1/r_3 Q} \left| W_3(\lambda, \chi_l \right|$$

$$\times \sum_{r_4 \leqq P} r_4^{-4/13+2\varepsilon} \sideset{}{^*}\sum_{\chi_4 \bmod r_4} \left(\int\limits_{-1/Qr_4}^{1/Qr_4} \left| W_4(\lambda, \chi_4) \right|^2 d\lambda \right)^{1/2}$$

$$\times \sum_{r_5 \leqq P} r_5^{-8/13+2\varepsilon} \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \left(\int\limits_{-1/Qr_5}^{1/Qr_5} \left| W_5(\lambda, \chi_5) \right|^2 d\lambda \right)^{1/2}$$

$$\ll L^c I_2 I_3 W_4 W_5,$$

where

$$I_k = \sum_{r \leqq P} r^{-a_k} \sideset{}{^*}\sum_{\chi} \max_{|\lambda| \leqq 1/rQ} \left| W_k(\lambda, \chi) \right|,$$

$$W_k = \sum_{r \leqq P} r^{-a_k} \sideset{}{^*}\sum_{\chi} \left(\int\limits_{-1/Qr}^{1/Qr} \left| W_k(\lambda, \chi) \right|^2 d\lambda \right)^{1/2},$$

$$a_k = \begin{cases} \varepsilon, & \text{for } k = 2, \\ \frac{1}{13} - 2\varepsilon, & \text{for } k = 3, \\ \frac{4}{13} - 2\varepsilon, & \text{for } k = 4, \\ \frac{8}{13} - 2\varepsilon, & \text{for } k = 5. \end{cases}$$

Arguing similarly we obtain

$$(3.7) \qquad S_1 + S_2 + S_3$$

$$\ll L^c \max_{\substack{2 \leqq k,l,m,n \leqq 5 \\ k+l+m+n=\mu+1}} \max_{|\lambda| \leqq 1/Q} \left|T_k(\lambda)\right| \max_{|\lambda| \leqq 1/Q} \left|T_l(\lambda)\right| \left(\int_{-1/Q}^{1/Q} \left|T_m(\lambda)\right|^2 d\lambda \right)^{1/2} W_n$$

$$+ L^c \max_{\substack{2 \leqq k,l,m,n \leqq 5 \\ k+l+m+n=\mu+1}} \max_{|\lambda| \leqq 1/Q} \left|T_k(\lambda)\right| \max_{|\lambda| \leqq 1/Q} \left|T_l(\lambda)\right| W_m W_n$$

$$+ L^c \max_{\substack{2 \leqq k,l,m,n \leqq 5 \\ k+l+m+n=\mu+1}} \max_{|\lambda| \leqq 1/Q} \left|T_k(\lambda)\right| I_l W_m W_n.$$

We have trivially

$$\max_{|\lambda| \leqq 1/Q} \left|T_k(\lambda)\right| \ll x^{1/k}.$$

Using (3.4) we obtain

$$\left(\int_{-1/Q}^{1/Q} \left|T_k(\lambda)\right|^2 d\lambda \right)^{1/2} \ll x^{\frac{1}{k} - \frac{1}{2}}.$$

Thus we see from (3.3) and (3.5)–(3.7) that the proof of (2.3) reduces to the proof of the following two lemmata:

LEMMA 3.5. *If* $P \leqq x^{\frac{13}{180} - \varepsilon}$ *and* $2 \leqq k \leqq 5$

$$W_k \ll_B x^{1/k - 1/2} L^{-B}$$

for any $B > 0$.

LEMMA 3.6. *If* $P \leqq x^{\frac{13}{180}-\varepsilon}$ *and* $2 \leqq k \leqq 5$

$$I_k \ll x^{1/k} L^A$$

for a certain $A > 0$.

4. Proof of lemma 3.5

In order to prove the lemma it is enough to show that

$$W_{k,R} \ll x^{\frac{1}{k}-\frac{1}{2}} R^{a_k} L^{-B}, \tag{4.1}$$

where

$$W_{k,R} = \sum_{r \sim R} \sum_{\chi}{}^{*} \left(\int_{-1/Qr}^{1/Qr} \left| W_k(\lambda, \chi) \right|^2 d\lambda \right)^{1/2}$$

for $R \leqq P/2$. Applying lemma 1, [3] we see

$$\int_{-1/Qr}^{1/Qr} \left| W_k(\lambda, \chi) \right|^2 d\lambda \tag{4.2}$$

$$\ll (QR)^{-2} \int_{x/2^{k+2}}^{x} \left| \sum_{\substack{t < m^k \leqq t+Qr \\ x/2^{k+1} < m^k \leqq x}} \Lambda(m)\chi(m) - E_0 \sum_{\substack{t < m^k \leqq t+Qr \\ x/2^{k+1} < m^k \leqq x}} 1 \right|^2 dt.$$

We set $X = \max(x/2^{k+1}, t)$ and $X + Y = \min(x, t + Qr)$. In the sequel we will treat the cases $R > L^D$ and $R \leqq L^D$ for a sufficiently large constant $D > 0$ separately. In the first case we apply a slight modification of Heath-Brown's identity ([4])

$$-\frac{\zeta'}{\zeta}(s) = \sum_{j=1}^{K} \binom{K}{j} (-1)^{j-1} \zeta'(s) \zeta^{j-1}(s) M^j(s) - \frac{\zeta'}{\zeta}(s) \bigl(1 - \zeta(s) M(s)\bigr)^K,$$

with $K = 5$ and

$$M(s) = \sum_{n \leqq x^{1/5k}} \mu(n)$$

to the sum

$$\sum_{X<m^k\leqq X+Y}.$$

Arguing exactly as in part III, [11] we find by applying Heath Brown's identity and Perron's summation formula (see [9], Lemma 3.12) that the inner sum of (4.2) — where always $E_0=0$ because of $R>L^D$ and the primitivity of the characters — is a linear combination of $O(L^c)$ terms of the form

$$S_{k,I_{a_1},\dots,I_{a_{10}}}=\frac{1}{2\pi i}\int_{-T}^{T}F_k\left(\frac{1}{2}+iu,\chi\right)\frac{(X+Y)^{\frac{1}{k}\left(\frac{1}{2}+iu\right)}-X^{\frac{1}{k}\left(\frac{1}{2}+iu\right)}}{\frac{1}{2}+iu}\,du$$

$$+O\big(T^{-1}x^{\frac{1}{k}+\varepsilon}\big),$$

where $2\leqq T\leqq x$,

$$F_k(s,\chi)=\prod_{j=1}^{10}f_{k,j}(s,\chi),\quad f_{k,j}(s,\chi)=\sum_{n\in I_{k,j}}a_{k,j}(n)\chi_n n^{-s},$$

$$a_{k,j}(n)=\left\{\begin{array}{ll}\log n\text{ or }1, & j=1,\\ 1, & 1<j\leqq 5,\\ \mu(n), & 6\leqq 10.\end{array}\right\},\quad I_j=(N_{k,j},\ 2N_{k,j}],\quad 1\leqq j\leqq 10,$$

$$\sqrt[k]{x}\ll\prod_{j=1}^{10}N_{k,j}\ll\sqrt[k]{x},\quad N_{k,j}\leqq x^{1/5k},\quad 6\leqq j\leqq 10. \tag{4.3}$$

Since

$$\frac{(X+Y)^{\frac{1}{k}\left(\frac{1}{2}+iu\right)}-X^{\frac{1}{k}\left(\frac{1}{2}+iu\right)}}{\frac{1}{2}+iu}\ll\min\left(QRx^{\frac{1}{2k}-1},x^{\frac{1}{2k}}\left(|u|+1\right)^{-1}\right)$$

by taking $T=x^{2\varepsilon}P^2\big(1+|\lambda|x\big)$ and $T_0=x(QR)^{-1}$, we conclude that $S_{I_{a_1},\dots,I_{a_{11}}}$ is bounded by

$$\ll QRx^{\frac{1}{2k}-1}\int_{-T_0}^{T_0}\left|F_k\left(\frac{1}{2}+iu,\chi\right)\right|du+x^{\frac{1}{2k}}\int_{T_0\leqq|u|\leqq T}\left|F_k\left(\frac{1}{2}+iu,\chi\right)\right|\frac{du}{|u|}$$

$$+\,x^{\frac{1}{k}}P^{-2},$$

Thus we derive from (4.2) that in order to prove (4.1) it is enough to show that

$$(4.4)\qquad \sum_{r\sim R}\ \sum_{\chi}{}^{*}\int_{0}^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|\,dt \ll x^{1/2k}R^{a_k-\varepsilon}L^{-B},$$

$$(4.5)\qquad \sum_{r\sim R}\ \sum_{\chi}{}^{*}\int_{T_1}^{2T_1}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|\,dt \ll x^{1/2k-1}QR^{1+a_k-\varepsilon}T_1L^{-B},$$

$$T_0<|T_1|\leqq T.$$

For the proof of (4.4) and (4.5) we will prove two propositions. We will need the estimate

$$(4.6)\qquad \sum_{n\leqq x} d^k(n)\ll_k xL^{c(k)}.$$

We now establish

PROPOSITION 1. *If there exists N_{k,j_1} and $N_{k,j2}$ $(1\leqq j_1,j_2\leqq 5)$ such that $N_{k,j1}N_{k,j2}\geqq P^{2-2a_k+3\varepsilon}$ then (4.4) is true.*

PROOF. We suppose without loss of generality $j_1=1$, $a_1(n)=\log n$ and $j_2=2$, $a_2(n)=1$. Arguing exactly as in the proof of proposition 1 in [11], we find

$$f_{k,1}\left(\frac{1}{2}+it,\chi\right)\ll L\left(\int_{-x^{1/k}}^{x^{1/k}}\left|L'\left(\frac{1}{2}+it+iv,\chi\right)\right|^4\frac{dv}{1+|v|}\right)^{1/4}+L,$$

and so we find by using lemma 3.7 in [2]:

$$\sum_{r\sim R}\ \sum_{\chi}{}^{*}\int_{0}^{T_0}\left|f_1\left(\frac{1}{2}+it,\chi\right)\right|^4dt$$

$$\ll L^4\int_{-x^{1/k}}^{x^{1/k}}\frac{dv}{1+|v|}\sum_{r\sim R}\ \sum_{\chi}{}^{*}\int_{v}^{T_0+v}\left|L'\left(\frac{1}{2}+it,\chi\right)\right|^4dt+T_0R^2L^4$$

$$\ll L^5 \max_{|N|\leq x^{1/k}} \int_{N/2}^{N} \frac{dv}{1+|v|} \sum_{r\sim R} \sum_{\chi}{}^{*} \int_{v}^{T_0+v} \left|L'\left(\frac{1}{2}+it,\chi\right)\right|^4 dt + T_0R^2L^4$$

$$+ L^5 \max_{|N|\leq x^{1/k}} N^{-1} \int_{0}^{T_0} dt \sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{\frac{N}{2}+t}^{N+t} \left|L'\left(\frac{1}{2}+iv,\chi\right)\right|^4 dv + T_0R^2L^4$$

$$\ll R^2T_0L^c,$$

Using lemma 3.8 in [2], (4.6) and Hölder's inequality we obtain

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{0}^{T_0} \left|F_k\left(\frac{1}{2}+it,\chi\right)\right| dt$$

$$\ll \left(\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{0}^{T_0} \left|f_{k,1}\left(\frac{1}{2}+it,\chi\right)\right| dt\right)^{1/4}$$

$$\times \left(\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{0}^{T_0} \left|f_{k,2}\left(\frac{1}{2}+it,\chi\right)\right| dt\right)^{1/4}$$

$$\times \left(\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{0}^{T_0} \left|\prod_{j=3}^{10} f_{k,j}\left(\frac{1}{2}+it,\chi\right)\right| dt\right)^{1/2}$$

$$\ll (R^2T_0)^{1/2} \left(R^2T_0 + \frac{x^{1/k}}{N_{k,1}N_{k,2}}\right)^{1/2} L^c \ll x^{1/2k}R^{a_k}L^{-B},$$

by the definition of T_0 and the condition of the proposition. □

PROPOSITION 2. *Let $J = \{1, \ldots, 10\}$. If J can be divided into two non overlapping subsets J_1 and J_2 such that*

$$\max\left(\prod_{j\in J_1} N_{k,j}, \prod_{j\in J_2} N_{k,j}\right) \ll x^{\frac{1}{k}}P^{-2+2a_k-3\varepsilon}$$

then (4.4) is true.

PROOF. Let

$$F_{k,i}(s,\chi)=\prod_{j\in J_i} f_{k,j}(s,\chi)=\sum_{n\ll M_i} b_i(n)\chi(n)n^{-s},\ \ b_i(n)\ll d^c(n),\ \ i=1,2,$$

where $M_i=\prod_{j\in J_i} N_{k,j}$, $i=1,2$. Applying lemma 3.8 in [2], (4.3) and (4.6) we see

$$\begin{aligned}
&\sum_{r\sim R}\ \sum_{\chi}{}^{*}\int_0^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt\\
&\ll\left(\sum_{r\sim R}\ \sum_{\chi}{}^{*}\int_0^{T_0}\left|F_{k,1}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/2}\\
&\times\left(\sum_{r\sim R}\ \sum_{\chi}{}^{*}\int_0^{T_0}\left|F_{k,2}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/2}\\
&\ll\left(R^2T_0+M_1\right)^{1/2}\left(R^2T_0+M_2\right)^{1/2}\\
&\ll R^2T_0+x^{\frac{1}{2k}}RP^{-1+a_k-\frac{3}{2}\varepsilon}T_0^{1/2}+x^{1/2k}L^c.
\end{aligned}$$

This proves the proposition because of $R>L^D$. Using proposition 1 and 2, we can prove (4.4) in nearly the same way as (4.4) is proved in [2]. The only difference in the proof is that instead of assuming

$$N_{k,i}N_{k,j}\leqq P^{12/7+3\varepsilon}\leqq x^{2/5k},\quad 1\leqq i,j\leqq 5,\quad i\neq j$$

as in [2], we assume in view of proposition 1 that

$$N_{k,i}N_{k,j}\leqq P^{2-2a_k+3\varepsilon}\leqq x^{2/5k},\quad 1\leqq i,j\leqq 5,\quad i\neq j.$$

The proof of (4.5) goes along the same lines. (4.1) is now proved in the case $R>L^D$. The case $R\leqq L^D$ is treated exactly as in [2]. □

5. Proof of Lemma 3.6

To prove the lemma it is enough to show that

$$\max_{R\leqq P/2}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\max_{|\lambda|\leqq 1/rQ}\left|W_k(\lambda,\chi_r)\right|\ll x^{1/k}R^{a_k}L^A.$$

Arguing as in the section before — we do not have to apply Gallagher's lemma here — we find

$$W_k(\lambda,\chi)\ll L^c\max_{I_{a_1},\dots,I_{a_{2k+1}}}\left|\int_{-T}^{T}F\left(\frac{1}{2}+it,\chi\right)dt\right.$$
$$\left.\times\int_{x/2^{k+1}}^{x}u^{\frac{1}{2k}-1}e\left(\frac{t}{2k\pi}\log u+\lambda u\right)du\right|+x^{1/k}P^{-1},$$

for $T=P^3$. Estimating the inner integral by the lemmata 4.3 and 4.4 of [9], we obtain

$$\int_{x/2^{k+1}}^{x}u^{\frac{1}{2k}-1}e\left(\frac{t}{2k\pi}\log u+\lambda u\right)du$$
$$\ll x^{\frac{1}{2k}-1}\min\left(\frac{x}{\sqrt{|t|+1}},\frac{x}{\min\limits_{x/2^{k+1}<u\leqq x}|t+2k\pi\lambda u|}\right).$$

Taking $T_0=4k\pi x(rQ)^{-1}$ we conclude that in order to prove the lemma it is enough to prove that for $P\leqq x^{\frac{7}{150}-\varepsilon}$ and $2\leqq k\leqq 5$ there holds

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_{0}^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt\ll x^{1/2k}R^{a_k}L^c, \tag{5.1}$$

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_{T_1}^{2T_1}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt\ll x^{1/2k}R^{a_k}T_1L^c,\quad T_0<|T_1|\leqq T. \tag{5.2}$$

These estimates are shown in the same way as (4.4) and (4.5). Two propositions analogous to the propositions 1 and 2 are proved:

PROPOSITION 3. *If there exist* N_{k,j_1} *and* $N_{k,j2}$ $(1 \leqq j_1, j_2 \leqq 5)$ *such that* $N_{k,j1}N_{k,j2} \geqq P^{2-2a_k+3\varepsilon}$ *then* (5.1) *is true.*

PROPOSITION 4. *Let* $J = \{1, \dots, 10\}$. *If* J *can be divided into two non overlapping subsets* J_1 *and* J_2 *such that*

$$\max\left(\prod_{j\in J_1} N_{k,j}, \prod_{j\in J_2} N_{k,j}\right) \ll x^{\frac{1}{k}} P^{-2+2a_k-3\varepsilon}$$

then (5.1) *is true.* □

REMARK. Here we do not need to treat the case $R > L^D$ separately because we do not have to save a factor L^{-B}.

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(Received August 21, 2002)

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