Studia Scientiarum Mathematicarum Hungarica 37 (2001), 1–20

ON GOLDBACH'S CONJECTURE IN ARITHMETIC PROGRESSIONS

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Communicated by I. Z. Ruzsa

Abstract

Let N be any odd positive integer and r and b_i positive integers with $(r, b_i) = 1$, $i \in \{1, 2, 3\}$. We prove that if $N \equiv b_1 + b_2 + b_3 \pmod r$, then there exists a constant $\delta > 0$ such that for any sufficiently large N and any $r \leqq N^\delta$ the equation

$$N = p_1 + p_2 + p_3$$

has prime solutions p_1, p_2 and p_3, which satisfy $p_i \equiv b_i \pmod r$.

1. Introduction and statement of results

The Goldbach–Vinogradov theorem states that every sufficiently large odd positive integer can be written as the sum of three primes. It also gives an asymptotic formula for the number of possible representations. Various generalizations of this problem have been investigated. Ayoub [1] proved the following unconditional result, which had been anticipated by Rademacher [8] assuming the Great Riemann Hypothesis: If r is a fixed positive integer, b_i $(i = 1, 2, 3)$ integers with $(b_i, r) = 1$ and $J(N; r, b_1, b_2, b_3)$ is the number of solutions of the equation

$$N = p_1 + p_2 + p_3, \quad p_j \equiv b_j \pmod r, \tag{1.1}$$

then

$$J(N; r, b_1, b_2, b_3) = \sigma(N; r) \frac{N^2}{2 \log^3 N} (1 + o(1)),$$

2000 *Mathematics Subject Classification.* Primary 11P32; Secondary 11L07.
Key words and phrases. Goldbach's problem, arithmetic progressions.

During the preparation of this article the author was staying at the Department of Mathematics at Shandong University, P.R. China. He was holding a common scholarship by the Chinese State Education Commission and the German Academic Exchange Service (DAAD).

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where for odd $N \equiv b_1 + b_2 + b_3 \pmod r$,

$$\sigma(N;r) = \frac{C(r)}{r^2} \prod_{p|r} \frac{p^3}{(p-1)^3+1} \prod_{\substack{p|N \\ p\nmid r}} \frac{(p-1)((p-1)^2-1)}{(p-1)^3+1} \times \prod_{p>2} \left(1 + \frac{1}{(p-1)^3}\right), \tag{1.2}$$

where all $p > 2$, $C(r) = 2$ for odd r and $C(r) = 8$ for even r. From Ayoub's article it is clear that his approach can also be used to prove (1.2) for all $r \leqq \log^A N$ for an arbitrary $A > 0$. Making use of the Siegel–Walfisz theorem, Ayoub's method cannot be applied to an r as large as $r = N^{\delta}$. Results for such larger moduli only exist so far for almost all moduli r. Wolke [9] first gave a proof for almost all prime moduli $r \leqq N^{\frac{1}{11}}$. This was improved by Liu and Zhan [4], who proved a similar result for almost all moduli r in a range even larger than that of Wolke. In this paper we will further contribute to this problem by showing the following theorem:

THEOREM. *There exists a computable constant $\delta^* > 0$ such that any sufficiently large odd integer N which satisfies $N \equiv b_1 + b_2 + b_3 \pmod r$ can be represented as in* (1.1) *for any*

$$r \leqq N^{\delta^*}. \tag{1.3}$$

To prove this theorem we apply a modification of Montgomery's and Vaughan's technique [7] established by Liu and Tsang in [6].

2. Notation and some preliminary results

For the proof of Theorem 1 we shall introduce the following notation. Let r be a fixed positive integer and q any positive integer. We set

$$r = r^* \prod_{i=1}^{k} p_i^{\alpha_i}, \qquad q = q^* \prod_{i=1}^{k} p_i^{\beta_i},$$

with $\left(q^* r^*, \prod_{i=1}^{k} p_i\right) = 1$ and $(q^*, r^*) = 1$. Define

$$h_{1q} = \prod_{\substack{i=1 \\ \alpha_i < \beta_i}}^{k} p^{\alpha_i}, \qquad h_{2q} = \prod_{\substack{i=1 \\ \alpha_i \geqq \beta_i}}^{k} p^{\beta_i}$$

(and an empty product equal to 1) such that

$$\left(\frac{r}{h_{1q}}, \frac{q}{h_{2q}}\right) = 1, \quad h_{1q}h_{2q} = h_q, \tag{2.1}$$

where

$$h_q = (r, q). \tag{2.2}$$

Then for $q = q_a q_b$ with $(q_a, q_b) = 1$

$$h_{iq} = h_{iq_a} h_{iq_b} \tag{2.3}$$

for $i \in \{1, 2\}$. For three integers b_1, b_2 and b_3 which satisfy $(b_i, r) = 1$ and three characters $\chi_i \bmod q_i'$, $q_i' | q$, $i \in \{1, 2, 3\}$ define

$$S_i(\alpha) = \sum_{\substack{\frac{N}{6} \le p < N \\ p \equiv b_i \pmod r}} \log p \, e(\alpha p), \qquad S(\chi_i, \alpha) = \sum_{\frac{N}{6} \le p < N} \chi_i(p) \log p \, e(\alpha p),$$

$$C(\chi_i, q, h, b_i, a) = \sum_{\substack{m=1 \\ m \equiv b_i \pmod h \\ (m,q)=1}}^{q} \chi_i(m) \, e\left(\frac{ma}{q}\right), \quad A(q, h_q, N, \chi_1, \chi_2, \chi_3) =$$

$$\frac{\phi^3(h_{1q})}{\phi^3(q/h_{2q})} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C(\chi_1, q, h_q, b_1, a) C(\chi_2, q, h_q, b_2, a) C(\chi_3, q, h_q, b_3, a) \, e\left(\frac{-aN}{q}\right).$$

The expression on the right side obviously also depends on the b_i but we do not indicate this dependence in the index because we will only argue for fixed b_i. Let $\chi_{0,q}$ denote the principal character modulo q and set

$$A(q, h_q, N, \chi_{0,q/h_{2q}}, \chi_{0,q/h_{2q}}, \chi_{0,q/h_{2q}}) = A(q, h_q, N),$$

$$R(N) = \sum_{\substack{\frac{N}{6} \le p_i < N \\ p \equiv b_i \pmod r \\ p_1 + p_2 + p_3 = N}} \log p_1 \, \log p_2 \, \log p_3, \tag{2.4}$$

$$L = \log N, \quad P = N^{\delta}, \quad Q = NP^{-5/4}, \quad T = P^c,$$

where δ is a sufficiently small chosen, positive constant and c is a large positive constant. Let $[a, b, c]$ denote the smallest common multiple of the three integers a, b and c.

It is a well-known fact (see [1]) that there is not more than one primitive character to a modulus $q \leqq T$ for which the corresponding L-function has a zero in the region

$$\sigma < 1 - \Theta(T), \quad |t| \leqq T, \quad \Theta(T) = \frac{c_1}{\log T}, \tag{2.5}$$

where c_1 is a small constant. If there is such an exceptional character, it is real and we denote it by $\tilde{\lambda}$ and its module by $\tilde{r}$. The corresponding exceptional zero is real, simple and unique and we denote it by $\tilde{\beta}$. If $\tilde{\lambda}$ exists, the zero-free region in (2.5) is extended to (see [3])

$$\Theta(T) = \frac{c_2}{\log T} \log\left(\frac{ec_1}{(1-\tilde{\beta})\log T}\right). \tag{2.6}$$

Furthermore, it is known that

$$\frac{c_3}{\tilde{r}^{1/2}\log^2 \tilde{r}} \leqq 1 - \tilde{\beta} \leqq \frac{c_1}{\log T}. \tag{2.7}$$

We want to ensure that

$$\tilde{r} \leqq P^{1/16}, \tag{2.8}$$

if $\tilde{r}$ exists. For this purpose it suffices that in (2.4) and (2.5) we take $P = N^{\delta/16c}$ instead of $P = N^{\delta}$ and $c_1/16c$ instead of c_1 if $\tilde{r} > P^{1/16}$. We also define

$$\Omega(T) = \begin{cases} (1-\tilde{\beta})\log T & \text{if } \tilde{\beta} \text{ exists,} \\ 1 & \text{otherwise,} \end{cases} \tag{2.9}$$

and

$$I(\alpha) = \int_{N/6}^{N} e(x\alpha)\,dx, \quad \tilde{I}(\alpha) = \int_{N/6}^{N} x^{\tilde{\beta}-1} e(x\alpha)\,dx,$$

$$I_{\chi}(\alpha) = \int_{N/6}^{N} e(x\alpha) \sum_{|\gamma|\leqq T}{}' x^{\rho-1}\, d, \quad S_{\chi} = (x,T) = \sum_{|\gamma|\leqq T}{}' x^{\beta-1},$$

where $\sum'_{|\gamma|\leqq T}$ denotes the summation over all zeros $= \beta + i\gamma$ of $L(s,\chi)$ lying inside the region: $|\gamma| \leqq T$, $\frac{1}{2} \leqq \beta \leqq 1 - \Theta(T)$, where $\Theta(T)$ is defined as in (2.6) or (2.5) depending on whether or not $\tilde{\beta}$ exists. Now we define the *major arcs* M and the *minor arcs* m as

$$M = \bigcup_{q \leqq P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right]$$

and

$$m = \left[-\frac{1}{Q}, 1 - \frac{1}{Q}\right].$$

Thus we arrive at

$$\begin{aligned} R(N) &= \int_M S_1(\alpha)S_2(\alpha)S_3(\alpha)e(-N\alpha)d\alpha \\ &\quad + \int_m S_1(\alpha)S_2(\alpha)S_3(\alpha)e(-N\alpha)d\alpha \\ &= R_1(N) + R_2(N). \end{aligned} \tag{2.10}$$

3. The contribution of the minor arcs

In the following paragraphs we shall always suppose α to be of the type

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leqq \frac{1}{qQ}, \quad 1 \leqq a \leqq q, \quad (a,q) = 1. \tag{3.1}$$

By applying Cauchy's inequality we find that

$$\begin{aligned} &\int_m S_1(\alpha)S_2(\alpha)S_3(\alpha)e(-n\alpha)\,d\alpha \\ &\ll \max_{\alpha \in m} |S_1(\alpha)| \left(\int_0^1 |S_2(\alpha)|^2\,d\alpha\right)^{1/2} \left(\int_0^1 |S_3(\alpha)|^2\,d\alpha\right)^{1/2} \\ &\ll \max_{\alpha \in m} |S_1(\alpha)| \frac{NL^2}{r}. \end{aligned} \tag{3.2}$$

In order to estimate the maximum we use the following theorem established by Balog und Perelli [2]: For $M \leqq N$, $(a,q) = 1$ and $h_q = (r,q)$,

$$\sum_{\substack{n \leqq M \\ n \equiv b (\mathrm{mod}\, r)}} \Lambda(n)\, e\left(\frac{a}{q} n\right) \ll L^3 \left(\frac{h_q N}{r q^{\frac{1}{2}}} + \frac{q^{\frac{1}{2}} N^{\frac{1}{2}}}{h_q^{\frac{1}{2}}} + \frac{N^{\frac{4}{5}}}{r^{\frac{2}{5}}}\right).$$

Applying partial summation and (3.1) we derive

$$\max_{\alpha\in m}|S_1(\alpha)| \ll (1+|\lambda|N)\max_{M\leq N}\left|\sum_{\substack{n\leq M\\ n\equiv b(\bmod r)}}\Lambda(n)\,e\Big(\frac{an}{q}\Big)\right|$$

$$\ll \left(1+\frac{N}{qQ}\right)\left(\frac{N}{q^{\frac{1}{2}}}+q^{\frac{1}{2}}N^{\frac{1}{2}}+\frac{N^{\frac{4}{5}}}{r^{\frac{2}{5}}}\right)L^3 \tag{3.3}$$

$$\ll \frac{NL^3}{P^{1/4}}.$$

From (3.2) and (3.3) follows

$$R_2(N)\ll\frac{N^2L^5}{rP^{1/4}}. \tag{3.4}$$

4. Some auxiliary results for the calculation of the integral over the major arcs

We quote the following lemma from [1]:

LEMMA 4.1. *Let* $(a,q)=1$ *and* $(b,h_q)=1$. *Then*

$$C(\chi_{0,q/h_{2q}},q,h_q,b,a)=\begin{cases}\mu(q/h_q)\,e\Big(\frac{tba}{h_q}\Big) & \text{if } (q/h_q,h_q)=1,\\ 0 & \text{otherwise,}\end{cases}$$

where $\frac{tq}{h_q}\equiv 1(\bmod r)$.

LEMMA 4.2. *Let* $\beta>\alpha\geqq 0$ *be two positive integers and* p *a fixed prime number. Then for a primitive character* χ *modulo* p^β, *an integer* a *with* $p|a$ *and* $(b,h_p)=1$ *there is*

$$C(\chi,p^\beta,p^\alpha,b,a)=0.$$

PROOF. If $\alpha=0$ the lemma is contained in Lemma 5.4, [7]. In the other case we write $a=gp$ and find

$$\begin{aligned}C(\chi,p^\beta,p^\alpha,b,a) &= e\Big(\frac{bg}{p^{\beta-1}}\Big)\sum_{s=1}^{p^{\beta-\alpha}}e\Big(\frac{sg}{p^{\beta-\alpha-1}}\Big)\chi(b+sp^\alpha)\\ &= e\Big(\frac{bg}{p^{\beta-1}}\Big)\sum_{v=1}^{p^{\beta-\alpha-1}}e\Big(\frac{vg}{p^{\beta-\alpha-1}}\Big)\sum_{w=1}^{p}\chi(b+vp^\alpha+wp^{\beta-1}).\end{aligned}$$

If $\beta-\alpha=1$ the inner sum in the first line vanishes and if $\beta-\alpha>1$ the inner sum in the second line vanishes because χ is primitive.

LEMMA 4.3. *Let $\beta>\alpha\geqq 0$ be two positive integers, p a fixed prime number and a an integer with $(p,a)=1$. For any character $\chi=\chi^*\chi_0$, where χ^* is a primitive character modulo p^γ with $1\leqq\gamma<\beta$ and χ_0 is the principal character modulo p^Δ with $\gamma\leqq\Delta\leqq\beta$, and for any $(b,h_p)=1$ we have*

$$C(\chi,p^\beta,p^\alpha,b,a)=0.$$

PROOF. If $\alpha=0$ this follows from Lemma 5.4 in [7]. In the other case, similar to the proof of Lemma 4.2, we obtain:

$$\begin{aligned}C(\chi,p^\beta,p^\alpha,b,a) &= e\left(\frac{ba}{p^\alpha}\right)\sum_{s=1}^{p^{\beta-\alpha}} e\left(\frac{sa}{p^{\beta-\alpha}}\right)\chi^*(b+sp^\alpha)\\ &= e\left(\frac{ba}{p^\alpha}\right)\sum_{v=1}^{p^{\beta-\alpha-1}} e\left(\frac{va}{p^{\beta-\alpha}}\right)\chi^*(b+vp^\alpha)\sum_{w=1}^{p} e\left(\frac{wa}{p}\right).\end{aligned}$$

For $\beta-\alpha=1$ the inner sum in the first line vanishes whereas for $\beta-\alpha>1$ the inner sum in the second line vanishes.

LEMMA 4.4. *Let r be a positive integer, q a positive integer with $q=q_1q_2$, $(q_1,q_2)=1$, b an integer with $(b,h_q)=1$ and $\chi_d(\mathrm{mod}\,q/h_{2q})$ three characters with $\chi_d(\mathrm{mod}\,q/h_{2q})=\chi_{d1}(\mathrm{mod}\,q_1/h_{2q_1})\times\chi_{d2}(\mathrm{mod}\,q_2/h_{2q_2})$ $(d\in\{a,b,c\})$.*

(a) *For $g=g_1q_2+g_2q_1$, $(g_1,q_1)=1$ and $(g_2,q_2)=1$ we have*

$$C(\chi_d,q,h_q,b,g)=C(\chi_{d1},q_1,h_{q_1},b,g_1)C(\chi_{d2},q_2,h_{q_2},b,g_2).$$

(b) $$A(q,h_q,N,\chi_a,\chi_b,\chi_c)=\prod_{i=1}^{2}A(q_i,h_{q_i},N,\chi_{ai},\chi_{bi},\chi_{ci}).$$

(c) *For a primitive character χ modulo q and an integer a with $(q/h_q,a)>1$ we have:*

$$C(\chi,q,h_q,b,a)=0.$$

PROOF. (a) By definition

$$C(\chi_d,q,h_q,b,g) = \sum_{\substack{m=1\\ m\equiv b\,(\mathrm{mod}\,h_q)\\ (m,q)=1}}^{q} \chi_d(m)\,e\left(\frac{mg}{q}\right). \tag{4.1}$$

We choose two integers b_1 modulo h_{q_1} and b_2 modulo h_{q_2} such that

$$b_1q_2 + b_2q_1 \equiv b \,(\mathrm{mod}\, h_q), \tag{4.2}$$

(these numbers exist because $(q_1, q_2) = 1$). Thus we can write any m over which is summed in (4.1) as $m = m_1q_2 + m_2q_1$ with $m_i \equiv b_i \,(\mathrm{mod}\, h_{q_i})$. From (4.2) we see that $b_i \equiv b\overline{q_j} \,(\mathrm{mod}\, h_{q_i})$, $(j \in \{1,2\}, j \neq i,\ q_j\overline{q_j} \equiv 1 \,(\mathrm{mod}\, h_{q_i}))$ and so from (4.1) we obtain

$$\begin{aligned}
&C(\chi_d, q, h_q, b, g) = \\
&= \sum_{\substack{m_1=1 \\ m_1 \equiv b\overline{q_2} \,(\mathrm{mod}\, h_{q_1}) \\ (m_1,q_1)=1}}^{q_1} \sum_{\substack{m_2=1 \\ m_2 \equiv b\overline{q_1} \,(\mathrm{mod}\, h_{q_2}) \\ (m_2,q_2)=1}}^{q_2} e\left(\frac{(m_1q_2+m_2q_1)(g_1q_2+g_2q_1)}{q_1q_2}\right) \\
&\times \chi_{d1}\chi_{d2}(m_1q_2+m_2q_1) \\
&= \sum_{\substack{m_1=1 \\ m_1 \equiv b\overline{q_2} \,(\mathrm{mod}\, h_{q_1}) \\ (m_1,q_1)=1}}^{q_1} e\left(\frac{m_1g_1q_2}{q_1}\right)\chi_{d1}(m_1q_2) \sum_{\substack{m_2=1 \\ m_2 \equiv b\overline{q_1} \,(\mathrm{mod}\, h_{q_2}) \\ (m_2,q_2)=1}}^{q_2} e\left(\frac{m_2g_2q_1}{q_2}\right)\chi_{d2}(m_2q_1) \\
&= \sum_{\substack{m_1=1 \\ m_1 \equiv b \,(\mathrm{mod}\, h_{q_1}) \\ (m_1,q_1)=1}}^{q_1} e\left(\frac{m_1g_1}{q_1}\right)\chi_{d1}(m_1) \sum_{\substack{m_2=1 \\ m_2 \equiv b \,(\mathrm{mod}\, h_{q_2}) \\ (m_2,q_2)=1}}^{q_2} e\left(\frac{m_2g_2}{q_2}\right)\chi_{d2}(m_2) \\
&= C(\chi_{d1}, q_1, h_{q_1}, b, g_1)C(\chi_{d2}, q_2, h_{q_2}, b, g_2).
\end{aligned}$$

Part (b) follows from Part (a) and (2.3). For the proof of Part (c) we note that there exists a $h_{p^\alpha} < p^\alpha \parallel q$, $\alpha \geqq 1$ satisfying $p|a$. We write $a = a_2p^\alpha + a_1\dfrac{q}{p^\alpha}$ and see from Part (a) that it is enough to show that $C(., p^\alpha, \ldots, a_1) = 0$. From $p|a$ follows $p|a_1$ and therefore Part (c) follows from Lemma 4.2.

LEMMA 4.5. *For any fixed positive integer N that satisfies the congruence conditions of Theorem 1, an integer q and integers $(b_i, r) = 1$, $i \in \{1,2,3\}$ define:*

$$\begin{aligned}
N(q) = \mathrm{card}\Big\{&a_1, a_2, a_3 : 1 \leqq a_i \leqq q,\ a_i \equiv b_i \,(\mathrm{mod}\, h_q),\ (a_i, q) = 1, \\
&(i \in \{1,2,3\}),\ a_1 + a_2 + a_3 \equiv N \,(\mathrm{mod}\, q)\Big\}.
\end{aligned}$$

(a) *For $(q_1, q_2) = 1$ we have*

$$N(q_1q_2) = N(q_1)N(q_2).$$

(b) *Let* $p^\alpha \| r$. *For every integer* $\beta \geqq \alpha + 1 \geqq 2$ *we have*

$$\frac{p^\beta \phi^3(h_{1p^\beta})}{\phi^3\left(\dfrac{p^\beta}{h_{2p^\beta}}\right)} N(p^\beta) = \frac{p^\alpha \phi^3(h_{1p^\alpha})}{\phi^3\left(\dfrac{p^\alpha}{h_{2p^\alpha}}\right)} N(p^\alpha) = p^\alpha. \tag{4.3}$$

For $\beta \geqq \alpha + 1 = 1$ *we have*

$$\frac{p^\beta \phi^3(h_{1p^\beta})}{\phi^3\left(\dfrac{p^\beta}{h_{2p^\beta}}\right)} N(p^\beta) = \frac{p \phi^3(h_{1p})}{\phi^3\left(\dfrac{p}{h_{2p}}\right)} N(p) = \frac{pN(p)}{(p-1)^3}. \tag{4.4}$$

(c) *For* $p^\alpha \| r$ *and*

$$B(p) = \sum_{\beta \geqq 1} A(p^\beta, h_{p^\beta}, N), \tag{4.5}$$

we have

$$1 + B(p) = \begin{cases} \dfrac{p}{(p-1)^3} N(p) & \text{if } h_p = 1, \\ p^\alpha & \text{if } h_p > 1, \end{cases}$$

for $p^\alpha \| r$. *For* $(p, r) = 1$

$$1 + B(p) = 1 - \frac{c_p(-N)}{(p-1)^3},$$

is true with

$$c_q(N) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{aN}{q}\right).$$

Proof. (a) This can be shown in the same way as Lemma 4.4 (b) if we note that

$$N(q) = \frac{1}{q} \sum_{a=1}^{q} C(\chi_{0,q/h_{2q}}, q, h_q, b_1, a) C(\chi_{0,q/h_{2q}}, q, h_q, b_2, a) \times C(\chi_{0,q/h_{2q}}, q, h_q, b_3, a)\, e\left(\frac{-aN}{q}\right)$$

because of Lemma 4.4 (c).

(b) We write $a_i = b_i + c_i p^\alpha$ and $b_1 + b_2 + b_3 \equiv N \pmod{p^\alpha}$. We see that for $\beta \geqq \alpha + 1 \geqq 2$:

$$N(p^\beta) = \operatorname{card}\Big\{1 \leqq c_i \leqq p^{\beta-\alpha},\ (i \in \{1,2\}),\ (b_1 + b_2 + b_2 - N)/p^\alpha + c_1 + c_2 + c_2 \\ \equiv 0 \pmod{p^{\beta-\alpha}}\Big\} = p^{2(\beta-\alpha)}.$$

Taking into account $h_{1p^\beta} = h_{2p^\alpha} = p^\alpha$, $h_{2p^\beta} = h_{1p^\alpha} = 1$ for $\beta \geqq \alpha + 1$ the first equation in (4.3) follows straight and the second follows from $N(p^\alpha) = 1$. For the proof of (4.4) we note

$$N(p^\beta) = \operatorname{card}\Big\{a_1, a_2 : 1 \leqq a_i \leqq p^\beta,\ (a_i, p) = 1,\ (i \in \{1,2\}), \\ a_1 + a_2 - N \not\equiv 0 \pmod{p}\Big\}.$$

We write $a_i = c_i + d_i p^{\beta-1}$, $(i \in \{1,2\})$ and find

$$N(p^\beta) = \operatorname{card}\Big\{c_1, c_2, d_1, d_2 : 1 \leqq c_i \leqq p^{\beta-1},\ (c_i, p) = 1,\ (i \in \{1,2\}), \\ c_1 + c_2 - N \not\equiv 0 \pmod{p}, 1 \leqq d_i \leqq p\Big\} \\ = p^2 N(p^{\beta-1}).$$

Using $h_p = 1$ we can derive (4.4) by repeatedly applying the same argument.

(c) In the first case the Lemmas 4.1 and 4.4 (c) imply that $1 + B(p)$ is equal to

$$1 + A(p, h_p, n) = (p-1)^{-3} \sum_{h=1}^{p} \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \sum_{a_3=1}^{p-1} e\left(h \frac{a_1 + a_2 + a_3 - N}{p}\right) \\ = \frac{p}{(p-1)^3} N(p).$$

In the other case we have

$$1 + \sum_{\beta \geqq 1} A(p^\beta, h_{p^\beta}, N) = 1 + \sum_{\beta=1}^{\alpha} \phi(p^\beta) = 1 + \sum_{\beta=1}^{\alpha} \left(p^\beta - p^{\beta-1}\right) = p^\alpha. \tag{4.6}$$

The last part of the lemma follows straight from the definition.

Lemma 4.6.

$$\frac{1}{\phi^3(r)} \sum_{\substack{q \leqq P \\ (q, r_1) = 1}} A(q, h_q, N) = \sigma(N, r, r_1) + O((Pr)^{-1+\epsilon}), \tag{4.7}$$

with

$$\sigma(N,r,r_1) = \frac{1}{\phi^3(r)} \prod_{\substack{p^\beta \| r \\ (p,r_1)=1}} p^\beta \prod_{(p,rr_1)=1} \left(1 - \frac{c_p(-N)}{(p-1)^3}\right), \tag{4.8}$$

where $c_q(n)$ is defined as in Lemma 4.5 (c). Furthermore

$$\sigma(N,r,1) \gg \frac{r}{\phi^3(r)}. \tag{4.9}$$

PROOF. Applying Lemma 4.1 we can write the sum in (4.7) as

$$\frac{1}{\phi^3(r)} \sum_{\substack{q \leqq P \\ (q/h_q,h_q)=1 \\ (q,r_1)=1}} \cdots = \frac{1}{\phi^3(r)} \sum_{\substack{q \geqq 1 \\ (q/h_q,h_q)=1 \\ (q,r_1)=1}} \cdots + O\left(\frac{1}{\phi^3(r)} \sum_{\substack{q>P \\ (q/h_q,h_q)=1 \\ (q,r_1)=1}} \cdots\right).$$

It is known from [6] that for $r_1 = 1$ the first sum on the right side is absolutely convergent and equal to

$$\frac{C(r)}{r^2} \prod_{p|r} \frac{p^3}{(p-1)^3+1} \prod_{\substack{p|N \\ (p,r)=1}} \frac{(p-1)((p-1)^2-1)}{(p-1)^3+1} \prod_{p>2} \left(1 + \frac{1}{(p-1)^3}\right),$$

where all $p > 2$ and $C(r)$ is defined as in (1.2). A computation shows that this is equal to the right side in (4.8). For a general r_1 the same argument applies. Applying Lemma 4.1 term we get the following estimate for the error term uniformly for all r_1:

$$\begin{aligned}
\frac{1}{\phi^3(r)} \sum_{\substack{q>P,\,(q,r_1)=1 \\ (q/h_q,h_q)=1}} \cdots &\ll \frac{1}{\phi^3(r)} \sum_{\substack{q>P \\ (q/h_q,h_q)=1}} \frac{\phi(q)}{\phi^3\left(\frac{q}{h_q}\right)} \\
&= \frac{1}{\phi^3(r)} \sum_{\substack{q>P \\ (q/h_q,h_q)=1}} \frac{\phi(h_q)}{\phi^2\left(\frac{q}{h_q}\right)} \\
&\leqq \frac{1}{\phi^3(r)} \sum_{h|r} \phi(h) \sum_{\substack{(h,k)=1 \\ k \geqq P/h}} \frac{1}{\phi^2(k)} \\
&\ll_\epsilon \frac{1}{\phi^3(r)} P^{-1+\epsilon} \sum_{h|r} h^2 \\
&\leqq \frac{1}{\phi^3(r)} P^{-1+\epsilon} r^2 \zeta(2) \\
&\ll (Pr)^{-1+\epsilon},
\end{aligned}$$

where, in the last but one step, we have used the inequality $\sum_{d|r} d^2 \leqq r^2\zeta(2)$ which can be found in (3.44) in [8]. (4.9) follows from (4.8).

LEMMA 4.7. (a) *Let $\chi_1 \bmod k_1$, $\chi_2 \bmod k_2$ and $\chi_3 \bmod k_3$ be three primitive characters and $k=[k_1,k_2,k_3]$. Then*

$$\sum_{\substack{q\leqq P\\ q\equiv 0\ (\mathrm{mod}\ k)}} \left|A\left(q,h_q,N,\chi_1\chi_{0,\frac{q}{h_{2q}}},\chi_2\chi_{0,\frac{q}{h_{2q}}},\chi_3\chi_{0,\frac{q}{h_{2q}}}\right)\right| \ll \prod_p (1+B(p)),$$

where $B(p)$ is defined as in (4.5).

(b) *Let λ be a character. Then among all the $q\leqq 1$ there exists at most one q and at most one pair of characters $\chi_i\ (\mathrm{mod}\ q_i/h_{2q_i})$ and $\xi_i\ (\mathrm{mod}\ r/h_{1q_i})$ with $\xi_i\chi_i=\lambda_i$ and for which $C(\lambda,q,h_q,b,a)\neq 0$ for any $(b,r)=1$, $(a,q)=1$.*

PROOF. Denote J the left-hand side in the lemma. By the definition of the $C(\chi,\dots)$ we can obviously substitute $\chi_{0,\frac{q}{h_{2q}}}$ by $\chi_{0,q}$ in J. For each q over which is summed in J we write $q=q_aq_b$, where

$$(q_b,k)=1,\quad p|q_a\Longrightarrow p|k.$$

Therefore, by using Lemma 4.4 (b) we have

$$A(q,\dots)=A\left(q_a,h_{q_a},N,\chi_1\chi_{0,q_a},\chi_2\chi_{0,q_a},\chi_3\chi_{0,q_a}\right)A(q_b,h_{q_b},N). \tag{4.10}$$

From Lemmas 4.1, 4.3 and 4.4 (a) we obtain that

$$A(q_a,\dots)\neq 0\Longrightarrow q_a=k. \tag{4.11}$$

By using Lemmas 4.2, 4.4 (a) and 4.5 we see that

$$\begin{aligned}
&\left|A\left(k,h_k,N,\chi_1\chi_{0,k},\chi_2\chi_{0,k},\chi_3\chi_{0,k}\right)\right|\\
&= \left|\frac{k\phi^3(h_{1k})}{\phi^3\left(\frac{k}{h_{2k}}\right)}\sum_{\substack{1\leqq a_i\leqq k,\ (a_i,k)=1,\\ a_i\equiv b_i\ (\mathrm{mod}\ h_k)\\ a_1+a_2+a_3\equiv N\ (\mathrm{mod}\ k)}}\chi_1\chi_{0,k}(a_1)\chi_2\chi_{0,k}(a_2)\chi_3\chi_{0,k}(a_3)\right| \qquad (4.12)\\
&\leqq \frac{k\phi^3(h_{1k})}{\phi^3\left(\frac{k}{h_{2k}}\right)}N(k)\\
&= \prod_{p|k}(1+B(p)).
\end{aligned}$$

Moreover,

$$\prod_{(p,k)=1}\Big(1+\sum_{\alpha\geqq 1}|A(p^\alpha,h_{p^\alpha},N)|\Big)\ll\prod_{(p,k)=1}(1+B(p)). \tag{4.13}$$

The last inequality follows from Lemma 4.5 (c) and from an argument similar to its proof. Thus we derive Part (a) from (4.10), (4.12) and (4.13).

(b) Let $\xi_i\,(\mathrm{mod}\,\frac{r}{h_{1q_i}})$ and $\chi_i\,(\mathrm{mod}\,\frac{q_i}{h_{2q_i}})$ be any pair of characters with $\xi_i\chi_i=\lambda_i$. We write $q_i=q_i^*q$, where q is the largest divisor of q_i with $q/h_{2q}=1$ and $h_{1q}=1$. By using Lemma 4.4 (a) we have

$$C(\chi_i,q_i,h_{q_i},a,b_i)=C(\chi_i,q_i^*,h_{q_i^*},\dots,b_i)C(\chi_0,q,h_q,\dots,b_i).$$

Arguing as in (4.11) we can assume $h_{2q_i^*}=1$ and by applying $\xi_i\chi_i=\lambda_i$ we find:

$$K_i=q_i^*\frac{r}{h_{1q}}=q_i^{**}\prod_{i=l}^{m}p_l^{\beta_l}\frac{r}{\prod_{l=1}^{m}p_l^{\alpha_l}},$$

where

$$(q_i^{**},r)=1,\qquad \beta_l>\alpha_l>0,\quad p_l^{\alpha_l}\,\|\,r\quad\forall l. \tag{4.14}$$

If we suppose that there is another pair of characters $\xi_j\,(\mathrm{mod}\,\frac{r}{h_{1q_j}})$ and $\chi_j\,(\mathrm{mod}\,\frac{q_j}{h_{2q_j}})$ satisfying $\xi_j\chi_j=\lambda_i$ we see – when applying the corresponding notation – that

$$q_i^{**}\prod_{l=1}^{m}p_l^{\beta_l}\frac{r}{\prod_{l=1}^{m}p_l^{\alpha_l}}=q_j^{**}\prod_{g=1}^{n}(p_g^*)^{\beta_g}\frac{r}{\prod_{g=1}^{n}(p_g^*)^{\alpha_g}}.$$

From (4.14) we conclude that this equality holds only then if $q_i^*=q_j^*$ and if the products in both the numerators and the denominators are equal. This proves Part (b) of the lemma.

LEMMA 4.8. *If $x\geqq N^{1/2}$, there exists an absolute constant c_4 such that for a sufficiently small δ*

$$\sum_{q\leqq T}\ \sideset{}{^*}\sum_{\chi\,\mathrm{mod}\,q}S_\chi(x,T)\ll\Omega(T)^3\exp(-c_4/\delta),$$

where $\sideset{}{^*}\sum_{\chi\,\mathrm{mod}\,q}$ *denotes the summation over all primitive characters* $\chi\,\mathrm{mod}\,q$.

PROOF. This is Lemma 2.1 in [6].

5. Simplification of $\mathbf{I}_1(N)$

First we simplify the $S_i(\alpha)$. As in [4], we derive the following equivalence from (2.1) and the Chinese remainder theorem

$$\begin{matrix} x \equiv b_i (\text{mod } r) \\ x \equiv g\,(\text{mod } q) \end{matrix} \Longleftrightarrow \begin{matrix} x \equiv b_i (\text{mod } r/h_{1q}) \\ x \equiv g\,(\text{mod } q/h_{2q}). \end{matrix}$$

Thus we obtain

$$\begin{aligned} S_i\left(\frac{a}{q}+\eta\right) &= \sum_{\substack{g=1 \\ (g,q)=1 \\ g\equiv b_i\,(\text{mod } h_q)}}^{q} e\left(\frac{ga}{q}\right) \sum_{\substack{N/6\leq p<N \\ p\equiv b_i\,(\text{mod } r) \\ p\equiv g\,(\text{mod } q)}} \log p\, e(p\eta), \\ &= \sum_{\substack{g=1 \\ (g,q)=1 \\ g\equiv b_i\,(\text{mod } h_q)}}^{q} e\left(\frac{ga}{q}\right) \sum_{\substack{N/6\leq p<N \\ p\equiv b_i (\text{mod } r/h_{1q}) \\ p\equiv g (\text{mod } q/h_{2q})}} \log p\, e(p\eta). \end{aligned}$$

We shall introduce the Dirichlet characters ξ mod r/h_{1q} and χ mod q/h_{2q} and obtain

$$\begin{aligned} S_i\left(\frac{a}{q}+\eta\right) &= \frac{\phi(h_{1q})}{\phi(r)\phi(q/h_{2q})} \sum_{\xi \text{mod}\, r/h_{1q}} \bar{\xi}(b_i) \\ &\quad \times \sum_{\chi \text{mod}\, q/h_{2q}} C(\overline{\chi}, q, h_q, b_i, a) S(\xi\chi, \eta), \end{aligned} \tag{5.1}$$

where we have used the fact that the primes in the summation range are prime to the modules of all the characters $\xi\chi$. For further use we note that if $\xi\chi$ is induced by a primitive character λ^*, then

$$\lambda^* = \xi^* \chi^*, \tag{5.2}$$

for two primitive characters ξ^* and χ^*, whose moduli are prime to each other. For the exceptional Siegel character $\tilde{\lambda}$ to the module T we write

$$\tilde{\lambda} = \tilde{\xi}\tilde{\chi}, \quad \tilde{\xi} \,\text{mod}\, \tilde{r}_1, \quad \tilde{\chi} \,\text{mod}\, \tilde{r}_2 \tag{5.3}$$

with $\tilde{r} = \tilde{r}_1\tilde{r}_2$, where $\tilde{r}$ is the exceptional module and $\tilde{\xi}$ and $\tilde{\chi}$ are two real primitive characters. We know from Lemma 4.7 (b) that we only have to consider one pair $\tilde{\xi}$ and $\tilde{\chi}$. We now quote Lemma 3.1 in [6].

LEMMA 5.1. *For any real η and any character $\lambda \bmod q$ with $q \leqq P$, we have*

$$S(\lambda,\eta)=\delta_{\lambda_0}I(\eta)-\delta_{\tilde\lambda}I_{\tilde\lambda}(\eta)-I_\lambda(\eta)+O((1+|\eta|N)NL^2T^{-1}),$$

where

$$\delta_{\lambda_0}=\begin{cases}1 & \text{if } \lambda=\lambda_0 \pmod q,\\ 0 & \text{otherwise,}\end{cases}\qquad \delta_{\tilde\lambda}=\begin{cases}1 & \text{if } \lambda=\tilde\lambda\lambda_0 \pmod q,\\ 0 & \text{otherwise.}\end{cases}$$

First we simplify $I_1(N)$. Set

$$G_i(a,q,\eta)=\sum_{\xi \bmod r/h_{1q}}\overline{\xi}(b_i)\sum_{\chi \bmod q/h_{2q}}C(\overline{\chi},q,h_q,b_i,a)I_{\xi\chi}(\eta)$$

and

$$H_i(a,q,\eta)=C(\chi_{0,\frac{q}{h_{2q}}},q,h_q,b_i,a)I(\eta)$$
$$-\delta_q\tilde\xi(b_i)C(\tilde\chi\chi_{0,\frac{q}{h_{2q}}},q,h_q,b_i,a)\tilde I(\eta)-G_i(a,q,\eta),$$

where

$$\delta_q=\begin{cases}1 & \text{if } \tilde r_1|\frac{r}{h_{1q}},\ \tilde r_2|\frac{q}{h_{2q}},\\ 0 & \text{otherwise.}\end{cases}$$

For any $\alpha=\frac{a}{q}+\eta\in I(a,q)$ we use the argument on p. 604 in [6] and derive the following from (5.1):

$$S_i(\alpha)=\frac{\phi(h_{1q})}{\phi(r)\phi(q/h_{2q})}\Bigg(H_i(a,q,\eta)$$
$$+O\Bigg(\sum_{\xi \bmod r/h_{1q}}\sum_{\chi \bmod q/h_{2q}}(1+|\eta|N)|C(\overline{\chi},q,h_q,b_i,a)|NL^2T^{-1}\Bigg)\Bigg).$$

Using

$$\left|\sum_{\xi \bmod r/h_1}\sum_{\chi \bmod q/h_2}C(\overline{\chi},q,h_q,b_i,a)\right|\leqq\phi^2(q)\phi(r),$$

in the same way as in (3.17) in [6] we obtain:

$$I_1(N)=\frac{1}{\phi^3(r)}\sum_{q\leqq P}\frac{\phi^3(h_{1q})}{\phi^3(q/h_{2q})}$$
$$\times\sideset{}{^*}\sum_{a=1}^{q}e\left(-\frac{a}{q}N\right)\int_{-\infty}^{\infty}e(-\eta N)\prod_{i=1}^{3}H_i(a,q,\eta)+O(N^2P^{-1}r^{-2}),\tag{5.4}$$

for a sufficiently large c because of (1.3) and (2.4).

6. The major arcs

We first quote Lemma 4.7 and Lemma 7.2 (1) in [6]:

LEMMA 6.1. *For any complex numbers ρ_j with $0<\operatorname{Re}(\rho_j)\leqq 1$, $j=1,2,3$, we have*

$$\int_{-\infty}^{\infty} e(-N\eta)\left(\prod_{j=1}^{3}\int_{N/3}^{N} x^{\rho_j-1}e(\eta x)\,dx\right)d\eta = N^2\int_{\mathcal{D}}\prod_{j=1}^{3}(Nx_j)^{(\rho_j-1)}\,dx_1dx_2, \tag{6.1}$$

where

$$x_3=1-\sum_{j=1}^{2}x_j,$$

and

$$\mathcal{D}=\{(x_1,x_2):1/3\leqq x_1,x_2,x_3\leqq 1\}.$$

Further,

$$1\ll\int_{\mathcal{D}}1\,dx_1dx_2\ll 1. \tag{6.2}$$

We define

$$G(m_1,m_2,\dots):=\sum_{\substack{1\leqq l_i\leqq\tilde r_2,\ (l_i,\tilde r_2)=1\\ l_i\equiv b_i \ (\mathrm{mod}\ h_{\tilde r_2})\\ l_1+l_2+l_3\equiv N(\mathrm{mod}\ \tilde r_2)}}\tilde\chi(m_1)\tilde\chi(m_2)\dots,$$

$$P(m_1,m_2,\dots):=\int_{\mathcal{D}}(Nx_{m_1})^{\tilde\beta-1}(Nx_{m_2})^{\tilde\beta-1}\dots dx_1dx_2.$$

We know that $\prod_{i=1}^{3}H_i(a,q,n)$ is sum of 3^3 terms which can be divided into three groups:

T_1: the term $\prod_{i=1}^{3}C(\chi_{0,\frac{q}{h_{2q}}},q,h_q,b_i,a)$,

T_2: the 19 terms each of which has at least one $G_i(a,q,\eta)$ as a factor,

T_3: the remaining 7 terms.

We write for $i=1,2,3$

$$M_i = \frac{1}{\phi^3(r)} \sum_{q \leq P} \frac{\phi^3(h_{1q})}{\phi^3(q/h_{2q})} \sum_{a=1}^{q}{}^{*} e\left(\frac{-Na}{q}\right) \times \int_{-\infty}^{\infty} e(-N\eta)\{\text{sum of the terms in } T_i\}\, d\eta.$$

Thus we can write (5.4) as

$$I_1(N) = M_1 + M_2 + M_3 + O(N^2 P^{-1} r^{-2}). \tag{6.3}$$

Let

$$\mathcal{P}_0 = N^2 \int_{\mathcal{D}} 1\, dx_1 dx_2.$$

We now give lemmas concerning the contribution of the M_i. The proofs are the same as the ones of the Lemmas 5.1 to 6.1 in [6]. Therefore we will not give them in their full length. Our procedure differs far from the one in [6] in so far as we have assured in (2.8) that $\tilde{r}$ is small compared to P and T. So we need not prove an equivalent for Lemma 5.5 in [6].

LEMMA 6.2.

$$M_1 = \frac{1}{\phi^3(r)} \mathcal{P}_0 \prod_p (1 + B(p)) + O\left(N^2 (Pr)^{-1+\epsilon}\right).$$

This is proved in the same way as Lemma 5.1 in [6] by using Lemmas 4.5 (c), 4.6 and 6.1. With the notation of (5.3) we have:

LEMMA 6.3.

$$\begin{aligned} M_3 = \frac{1}{\phi^3(r)} N^2 \frac{\tilde{r}_2 \phi^3(h_{1\tilde{r}_2})}{\phi^3(\tilde{r}_2/h_{2\tilde{r}_2})} \prod_{(p,\tilde{r}_2)=1} (1+B(p)) \Bigg\{ &- \sum_{j=1}^{3} \tilde{\xi}(b_j) G(j) P(j) r \\ &+ \sum_{1 \leq i < j \leq 3} \tilde{\xi}(b_i)\tilde{\xi}(b_j) G(i,j) P(i,j) \\ &- \tilde{\xi}(b_1)\tilde{\xi}(b_2)\tilde{\xi}(b_3) G(1,2,3) P(1,2,3) \Bigg\} \\ &+ O(N^2 (P/\tilde{r}_2)^{-1+\epsilon} r^{\epsilon}). \end{aligned}$$

The treatment of the integral follows exactly the proof of Lemma 5.2 in [6] by applying Lemma 6.1. As an example for the treatment of the singular series we choose the term containing $G(1)$. Denoting the singular series W and arguing as in (4.10), (4.11) and (4.12), we obtain

$$W = \frac{1}{\phi^3(r)} \frac{\tilde{r}_2 \phi^3(h_{1\tilde{r}_2}) \tilde{\xi}(b_1)}{\phi^3(\tilde{r}_2/h_{2\tilde{r}_2})} G(1) \Big\{ \sum_{\substack{q \geqq 1 \\ (q,\tilde{r}_2)=1}} A(q, h_q, N) + O\Big(\sum_{q > P/\tilde{r}_2} |A(q,h_q,N)| \Big) \Big\},$$

where we have used that $\tilde{\xi}(b_1) = \tilde{\xi}\xi_0(b_1)$ for any principal character ξ_0 to a module of the form r/h_{1q}. Whereas the main term is already in the desired form due to Lemma 4.5 (c) and Lemma 4.6, we argue as in (4.13) and use Lemma 4.6 (b) to obtain

$$\frac{1}{\phi^3(r)} \frac{\tilde{r}_2 \phi^3(h_{1\tilde{r}_2})}{\phi^3(\tilde{r}/h_{2\tilde{r}_2})} |G(1)| \sum_{q > P/\tilde{r}_2} |A(q,h_q,N)| \ll \prod_{p | \tilde{r}_2} (1 + B(p))(Pr/\tilde{r}_2)^{-1+\epsilon}.$$

By applying Lemma 4.5 (c) we can estimate the product by $\ll r$ from which, together with (6.2), we can derive the error term. According to (1.3) we suppose $r \leqq N^{\delta^*} := T_1$ for a sufficiently small δ^* and we assume also $T^{1/2} \leqq T_1 \leqq T$. Following the lines of the proof of Lemma 5.3 in [6] we obtain from Lemmas 6.2 and 6.3:

Lemma 6.4.

$$M_1 + M_3 \gg \frac{\Omega^3(TT_1)}{\phi^3(r)} \mathcal{P}_0 \prod_p (1 + B(p)) + O(N^2 (P/\tilde{r}_2)^{-1+\epsilon} r^\epsilon).$$

Lemma 6.5.

$$M_2 \ll \frac{\Omega^3(TT_1)}{\phi^3(r)} \exp(-c_4/\delta) \mathcal{P}_0 \prod_p (1 + B(p)).$$

For the proof of Lemma 6.5 we treat exemplarily the term containing $G_i(a,q,\eta)$, $G_2(a,q,\eta)$ and $G_3(a,q,\eta)$. We denote it by L, apply Lemma 4.7 (b) and find:

$$L = \frac{1}{\phi^3(r)} \sum_{q \leqq P} \sum_{\xi_1 \bmod r/h_{1q}} \sum_{\chi_1 \bmod q/h_{2q}} \sum_{\xi_2 \bmod r/h_{1q}} \sum_{\chi_2 \bmod q/h_{2q}} \sum_{\xi_3 \bmod r/h_{1q}} \sum_{\chi_3 \bmod q/h_{2q}}$$

$$\times A(q, h_q, N, \chi_1, \chi_2, \chi_3) \int_{-\infty}^{\infty} e(-N\eta) I_{\xi_1\chi_1}(\eta) I_{\xi_2\chi_2}(\eta) I_{\xi_3\chi_3}(\eta)\, d\eta$$

$$= \int_{-\infty}^{\infty} e(-N\eta) \frac{1}{\phi^3(r)}$$

$$\times \sum_{q \leqq Pr} \sideset{}{'}\sum_{\lambda_1 \bmod q} \sideset{}{'}\sum_{\lambda_2 \bmod q} \sideset{}{'}\sum_{\lambda_3 \bmod q} I_{\lambda_1}(\eta) I_{\lambda_2}(\eta) I_{\lambda_3}(\eta) A(q, h_q, N, \chi_1, \chi_2, \chi_3),$$

where $\lambda_i = \xi_i \chi_i$ and $\sideset{}{'}\sum_{\lambda_i \bmod q}$ means that we only sum over such λ that can be written as $\lambda_i = \xi_i \chi_i$. Let χ_i be induced by $\chi_i^* \bmod r_i^*$ and λ_i be induced by $\lambda_i^* \bmod r_i$, then:

$$L \leqq \int_{-\infty}^{\infty} e(-N\eta) \frac{1}{\phi^3(r)} \sideset{}{'^*}\sum_{\substack{\lambda_1^* \bmod r_1 \\ r_1 \leqq Pr}} \sideset{}{'^*}\sum_{\substack{\lambda_2^* \bmod r_2 \\ r_2 \leqq Pr}} \sideset{}{'^*}\sum_{\substack{\lambda_3^* \bmod r_3 \\ r_3 \leqq Pr}} I_{\lambda_1^*}(\eta) I_{\lambda_2^*}(\eta) I_{\lambda_3^*}(\eta) d\eta$$

$$\sum_{\substack{q \leqq Pr \\ q \equiv 0 \bmod [r_1^*, r_2^*, r_3^*]}} |A(q, h_q, N, \chi_1^* \chi_{0,\frac{q}{h_{2q}}}, \chi_2^* \chi_{0,\frac{q}{h_{2q}}}, \chi_3^* \chi_{0,\frac{q}{h_{2q}}})|.$$

Now the lemma is proved in exactly the same way as Lemma 6.1 in [6] by using Lemmas 4.7, 4.8 (applied to TT_1 instead of T) and 6.1. From (6.3) and Lemmas 6.4 and 6.5 we derive for a sufficiently small δ:

$$I_1(N) \gg \frac{\Omega^3(TT_1)}{\phi^3(r)} \mathcal{P}_0 \prod_p (1 + B(p)) + O(N^2 (P/\tilde{r}_2)^{-1+\epsilon} r^\epsilon). \tag{6.4}$$

If $\tilde{\beta}$ exists, we can see from (2.7)–(2.9) that

$$\Omega(TT_1) = (1 - \tilde{\beta}) \log TT_1 \gg \tilde{r}^{-1/2} (\log \tilde{r})^{-1} \gg P^{-1/32} (\log \tilde{r})^{-1}. \tag{6.5}$$

We can now conclude the theorem from Lemma 4.5 (c), Lemma 4.6, (1.3), (2.4), (2.8), (2.10), (3.4), (4.9) and (6.2)–(6.5).

REMARK. The author wants to thank Dr. Jianya Liu for helpful discussions.

REFERENCES

[1] AYOUB, R., On Rademacher's extension of the Goldbach–Vinogradoff theorem, *Trans. Amer. Math. Soc.* **74** (1953), 482–491. *MR* **14**, 847a

[2] BALOG, A. and PERELLI, A., Exponential sums over primes in an arithmetic progression, *Proc. Amer. Math. Soc.* **93** (1985), 578–582. *MR* **86b**:11053

[3] GALLAGHER, P. X., A large sieve density estimate near $\sigma = 1$, *Invent. Math.* **11** (1970), 329–339. *MR* **43** #4775

[4] LIU, J. Y. and ZHAN, T., The ternary Goldbach problem in arithmetic progressions (to appear).

[5] LIU, M. C. and CHIU, S. F., On exceptional sets for numbers representable by binary sums (to appear).

[6] LIU, M. C. and TSANG, K. M., Small prime solutions of linear equations, *Théorie des nombres* (Quebec, PQ, 1987), W. de Gruyter, Berlin–New York, 1989, 595–624. *MR* **90i**:11112

[7] MONTGOMERY, H. L. and VAUGHAN, R. C., The exceptional set in Goldbach's problem, *Acta Arith.* **27** (1975), 353–370. *MR* **51** #10263

[8] RADEMACHER, H. A., Über eine Erweiterung des Goldbachschen Problems, *Math. Z.* **25** (1926), 627–657. *JFM* **52**.0167.04

[9] WOLKE D., Some applications to zero-density theorems for L-functions, *Acta Math. Hungar.* **61** (1993), 241–258. *MR* **94e**:11108

(Received May 23, 1998)

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