Throughput and delay bounds for input buffered switches using maximal weight matching algorithms and a speedup of less than two

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Abstract. Two main performance features of high-bandwidth switches are stability and delay. This paper investigates these performance features for input buffered switch architectures that deploy maximal weight matching algorithms to determine the configurations of the switching core. For this purpose, a novel mathematical model of the dynamics of a maximal weight matching algorithm is developed. Based on this model, it is shown that certain maximal weight matching algorithms provide stability when the switching core runs at a speedup of less than two. In addition, bounds on the expected average delay and on the absolute delay are established.

1 Introduction and Motivation

The core of most existing IP routers is based on a cell-based switching fabric. Because output queued switches become increasingly impractical due to the high speedup required in the switching core, most switches are based on either a pure input (IQ) or a combined input and output (CIOQ) buffered architecture. A typical CIOQ $N \times N$ switch is shown in figure 1. For each input i, there are N virtual output queues $VOQ_{i,j}$, $1 \leq j \leq N$. The cells arriving at input i and destined for output j are buffered in $VOQ_{i,j}$. The switching core itself is modeled as a crossbar requiring that not more than one packet can be sent simultaneously from the same input or to the same output. It works with a speedup of $S, S \geq 1$, i.e., it works at a speed S times faster than the speed of the input links. The choice of the scheduling algorithm is a major design criteria for switches. The scheduling algorithm should optimally provide guarantees on throughput and on average or absolute delay. In [4], [6] and [7], it has been shown that for a speedup of $S = 1$ there exist scheduling algorithms that provide guarantees for the throughput and the average delay of a switch. These algorithms solve the scheduling problem by finding a maximum weight matching of a bipartite $N \times N$ graph. The weights are chosen to be either the queue length of the $VOQs$ or the actual waiting time experienced by the head of line cells of the $VOQs$. Let $\lambda_{i,j}$ define the arrival rate of cells for $VOQ_{i,j}$. Traffic is said to be admissible if

$$\sum_{j=1}^{N} \lambda_{i,j} \leq 1, \qquad \sum_{i=1}^{N} \lambda_{i,j} \leq 1, \quad \forall\, i, j,\, 1 \leq i, j \leq N. \tag{1}$$

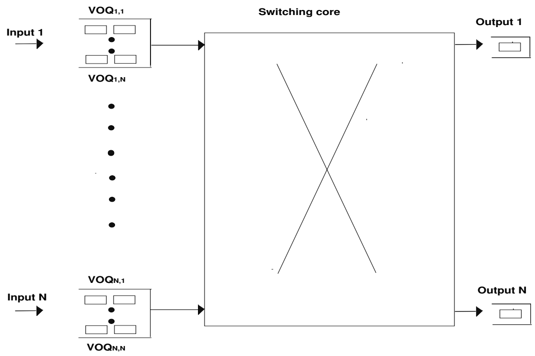

Fig. 1. Architecture of an input queued switch

However, the implementation of maximum weight algorithms is impractical as they have a complexity of $O(N^3 \log N)$. Therefore, the less complex class of maximal weight matching algorithms has been widely researched ([2], [8]). A maximal weight matching algorithm is defined for a set of weights $Q_{i,j}$, $1 \leq i,j \leq N$, where $Q_{i,j}$ is the weight assigned to $VOQ_{i,j}$, as follows:

1. Initially, all $VOQ_{i,j}$ are considered potential choices for a cell transfer.
2. The VOQ with the largest weight, say $VOQ_{a,b}$, is chosen for a cell transfer and ties are broken randomly.
3. All $VOQ_{i,j}$ with either $i = a$ or $j = b$ are removed.
4. If all $VOQ_{i,j}$ are removed, the algorithm terminates. Else go to step 2.

It has been shown in [3] and [5] that under the assumption of admissible traffic, every maximal weight matching algorithm deployed with a speedup of two guarantees the stability of the switch. In [1] these results were improved by showing that a switch that deploys any maximal weight matching algorithm is stable even with a speedup $S \geq R$, where

$$R = \max_{i,j} \left(\lambda_{i,j} + \sum_{k \neq j} \lambda_{i,k} + \sum_{l \neq i} \lambda_{l,j} \right) . \tag{2}$$

We note that due to (1), there holds $R < 2$. Under the additional assumption that ties are broken in a specific way, the stability of the $MM-LQF$ algorithm, the maximal weight matching algorithm with the weights chosen as the VOQ length, was even shown for $S = 1$ in [8]. (The $MM-LQF$ is denoted as $iLQF$ in [8].) Both papers describe the switch behavior via an approximate fluid model.

In contrast, this paper uses a discrete model to provide an analysis of switches deploying the $MM-LQF$ or the $MM-OCF$ algorithm, which uses the actual waiting time of the head of line cells as the weights, with a speedup $S > R$. It develops a new model to describe the dynamics of maximal matching algorithms.

Based on this model, we derive results on stability and - in contrast to the work quoted above - also bounds on the expected average delay a cell experiences at the input buffer. In addition, an absolute delay bound for the $MM - OCF$ algorithm deployed with a speedup $S > R$ is established for the first time.

The rest of the paper is organized as follows. Section 2 introduces the terminology to describe the dynamics of a switch and gives some preliminary results. In section 3, an inequality that characterizes the behavior of maximal weight matching algorithms is derived. In the sections 4 - 6, results on stability, the average and the absolute delay for the $MM - LQF$ and/or the $MM - OCF$ algorithms are proven. Our conclusions are presented in section 7.

2 Terminology and Model

Throughout this paper, the time t is described via a discrete, slotted time model. Cells are supposed to be of fixed size. An external timeslot is the time needed by a cell to arrive completely at an incoming link. As the switching core works at a speedup $S \geq 1$, an internal timeslot is defined as the time needed to transfer a cell through the switching core from an input to an output. Thus, the external timeslot from time t to $t+1$ consists of the S internal timeslots $[t+(k-1)/S, t+k/S]$, $1 \leq k \leq S$. For the sake of simplicity, we always assume that S is an integer. All proofs in this paper can be easily generalized for non-integer S by considering the dynamics of the switch over g external timeslots, where gS is an integer, instead of over one external timeslot as done in this paper. We suppose that cells arrive at the beginning of an external timeslot t and are transferred instantly at the end of an internal timeslot. We abbreviate $\sum_{i,j} = \sum_{1 \leq i,j \leq N}$ and define the norm of an $N \times N$ matrix as $||x||_1 = \sum_{i,j} x_{i,j}$. We define the arrival matrix $A(t)$, representing the arrivals at each VOQ:

$$A_{i,j}(t) = \begin{Bmatrix} 1, \text{ if an arrival at } VOQ_{i,j} \text{ occurs at at time } t, \\ 0, \text{ else.} \end{Bmatrix}$$

As there cannot arrive more than one cell at an input during an external timeslot, there holds

$$\sum_{1 \leq j \leq N} A_{i,j}(t) \leq 1, \qquad \forall i,\, 1 \leq i \leq N,\, \forall t. \tag{3}$$

The service matrix $S^k(t)$, indicating which queues are served at the end of the k-th internal timeslot of the t-th external timeslot is defined as:

$$S_{i,j}^k(t) = \begin{Bmatrix} 1, \text{ if } VOQ_{i,j} \text{ is served at the end of the } k\text{-th internal time slot} \\ \quad \text{of the } t\text{-th external time slot,} \\ 0, \text{ else.} \end{Bmatrix}$$

Due to the crossbar structure of the switch, the following inequalities hold:

$$\sum_{1 \leq i \leq N} S_{i,j}^k(t) \leq 1, \quad \sum_{1 \leq j \leq N} S_{i,j}^k(t) \leq 1, \;\; \forall i,j,\, 1 \leq i,j \leq N,\, \forall k,\, 1 \leq k \leq S, \forall t. \tag{4}$$

Now, we formalize the notion of the stability of a switch and give a sufficient criterion for stability which we will use later to establish some of our results.

Definition: Let $Y_n = (y_n(1), ..., y_n(M))$ be the row vector of a system of M queues at time n, where $y_n(i)$ is the length of the queue i at time n. A system of queues is said to be strongly stable if, for every $\epsilon > 0$, there exists $B > 0$ such that $\lim_{n\to\infty} P\{||Y_n||_1 > B\} < \epsilon$, and $\lim_{n\to\infty} \sup E||Y_n||_1 < \infty$.

Theorem 1 *Given a system of queues whose evolution is described by a discrete time Markov chain with state vector $X_n \in \mathbb{N}^M$, if a lower bounded function $V(X_n)$, called Lyapunov function, $V : \mathbb{N}^M \to \mathbb{R}$ can be found such that*

$$E[V(X_{n+1})|X_n] < \infty \; \forall \, X_n,$$

and there exist $\epsilon \in \mathbb{R}^+$ and $B \in \mathbb{R}^+$ such that

$$E[V(X_{n+1}) - V(X_n)|X_n] < -\epsilon \;\; \forall ||X_n|| > B,$$

then the system of queues is strongly stable.

Proof: This is a special instance of theorem 1 in [4].

3 A characterization of maximal weight matching algorithms

In this section, we develop an inequality that describes the dynamics of a maximal weight matching algorithm. We define $Q(t) = (Q_{1,1}(t), ..., Q_{N,N}(t))$ as the weights at the beginning of the t-th external timeslot and $Q^k_{i,j}(t)$, $1 \le k \le S$, as the weight of $VOQ_{i,j}$ at the beginning of the k-th internal timeslot of the t-th external timeslot. Thus, $Q_{i,j}(t) = Q^1_{i,j}(t)$, $\forall \, i, j$, $1 \le i, j \le N$. We show:

Theorem 2: *For any input buffered switch that applies a maximal weight matching algorithm and a speedup-up S, there holds at any time t*

$$\frac{1}{R} \sum_{k=1}^{S} \sum_{i,j} Q^k_{i,j}(t)\lambda_{i,j} \le \sum_{i,j} \sum_{k=1}^{S} Q^k_{i,j}(t) S^k_{i,j}(t).$$

Proof: In the first internal timeslot, in its first iteration, the algorithm selects the queue with the largest weight, say $Q^1_{a_1,b_1}$, for transfer. Thus, from (2):

$$\begin{aligned} Q^1_{a_1,b_1}(t)R &\ge Q^1_{a_1,b_1}(t) \left[\lambda_{a_1,b_1} + \sum_{k\neq b_1} \lambda_{a_1,k} + \sum_{l\neq a_1} \lambda_{l,b_1} \right] \\ &\ge Q^1_{a_1,b_1}(t)\lambda_{a_1,b_1} + \sum_{k\neq b_1} Q^1_{a_1,k}(t)\lambda_{a_1,k} + \sum_{l\neq a_1} Q^1_{l,b_1}(t)\lambda_{l,b_1}. \end{aligned}$$

All queues with either $i = a_1$ or $j = b_1$ are removed. In the second iteration, the remaining queue with the largest weight is chosen. Thus,

$$Q^1_{a_2,b_2}(t)R \ge Q^1_{a_2,b_2}(t)\lambda_{a_2,b_2} + \sum_{k\neq b_1,b_2} Q^1_{a_2,k}(t)\lambda_{a_2,k} + \sum_{l\neq a_1,a_2} \lambda_{l,b_2} Q^1_{l,b_2}(t). \quad (5)$$

The matching algorithm stops after k, $k \leq N$ iterations when only empty queues remain. For each of these k iterations, an inequality analogous to (5) holds. For the remaining empty queues, additional $(N-k)$ inequalities as in (5) hold where both sides are equal to zero. Summing over all N inequalities, we obtain

$$\sum_{m=1}^{N} Q^1_{a_m,b_m}(t) = \sum_{i,j} Q^1_{i,j}(t) S_{i,j}(t) \geq R^{-1} \sum_{i,j} Q^1_{i,j}(t) \lambda_{i,j}. \tag{6}$$

Applying this analysis to all S internal timeslots, we get the theorem.

4 Stability

In this section, we use the result of the previous section to prove stability for the $MM-LQF$ and $MM-OCF$ algorithms.

We set $L(t) = (L_{1,1}(n), .., L_{N,N}(t))$ where $L_{i,j}(t)$ defines the occupancy of $VOQ_{i,j}$ at the beginning of t-th external timeslot. We define $L^k_{i,j}(t)$, $1 \leq k \leq S$ as the occupancy of $VOQ_{i,j}$ at the beginning of the k-th internal timeslot in the t-th external timeslot. Therefore, $L_{i,j}(t) = L^1_{i,j}(t)$, $\forall\, i, j$, $1 \leq i, j \leq N$. The development of the VOQ occupancy between consecutive internal timeslots is described by the following equations:

$$L^k_{i,j}(t) = [L^{k-1}_{i,j}(t) - S^{k-1}_{i,j}(t)]^+, \ \forall k, \, 2 \leq k \leq S, \tag{7}$$

$$L^1_{i,j}(t) = [L^S_{i,j}(t-1) - S^S_{i,j}(t-1)]^+ + A_{i,j}(t), \tag{8}$$

where $[a]^+ = \max(0, a)$. Combining (7) and (8), we see:

$$L_{i,j}(t+1) = [L_{i,j}(t) - \sum_{k=1}^{S} S^k_{i,j}(t)]^+ + A_{i,j}(t+1).$$

For technical reasons, we also introduce the approximate next-state vectors:

$$\tilde{L}^k_{i,j}(t) = L^{k-1}_{i,j}(t) - S^{k-1}_{i,j}(t), \ \forall k, \, 2 \leq k \leq S, \tag{9}$$

$$\tilde{L}^1_{i,j}(t) = L^S_{i,j}(t-1) - S^S_{i,j}(t-1) + A_{i,j}(t), \tag{10}$$

$$\tilde{L}_{i,j}(t+1) = L_{i,j}(t) - \sum_{k=1}^{S} S^k_{i,j}(t) + A_{i,j}(t+1). \tag{11}$$

We see from (4) and (7):

$$L^S_{i,j}(t) \leq L^k_{i,j}(t) \ \forall k, \, 1 \leq k \leq S, \tag{12}$$

$$||L^S(t)||_1 \geq ||L(t)||_1 - (S-1)N. \tag{13}$$

Now we prove theorem 3 which was shown via a different approach in [1]:
Theorem 3: *The $MM-LQF$ algorithm with a speedup $S > R$ is stable for all admissible i.i.d. arrival processes.*

Proof: We define the Lyapunov function as $V(L(t)) = \sum_{i,j} L^2_{i,j}(t)$. We will give an upper bound for the expected value of the expression $V(\tilde{L}(t+1)) - V(L(t))$. By (7) and (9) there is, $V(\tilde{L}^k(t)) \geq V(L^k(t))$. Thus, we get from (9), (10), and (4):

$$\begin{aligned}
& V(\tilde{L}(t+1)) - V(L(t)) \\
&= V(\tilde{L}(t+1)) - V(L^S(t)) + \sum_{k=2}^{S} \left[V(L^k(t)) - V(L^{k-1}(t))\right] \\
&\leq V(\tilde{L}(t+1)) - V(L^S(t)) + \sum_{k=2}^{S} \left[V(\tilde{L}^k(t)) - V(L^{k-1}(t))\right] \\
&= 2\sum_{i,j} \left(A_{i,j}(t+1) - S^S_{i,j}(t)\right) L^S_{i,j}(t) + \sum_{i,j} \left(A_{i,j}(t+1) - S^S_{i,j}(t)\right)^2 \\
&\quad + \sum_{k=1}^{S-1} \left[-2\sum_{i,j} S^k_{i,j}(t) L^k_{i,j} + \sum_{i,j} \left(S^k_{i,j}(t)\right)^2\right]. \qquad (14)
\end{aligned}$$

Rearranging the last two sums $\sum_{i,j} .. + \sum_{k=1}^{S-1} ..$ in (14), we obtain from (3) and (4):

$$\sum_{i,j} A^2_{i,j}(t+1) - 2\sum_{i,j} A_{i,j}(t+1) S^S_{i,j}(t) + \sum_{k=1}^{S}\sum_{i,j} \left(S^k_{i,j}(t)\right)^2 \leq N(S+1). \qquad (15)$$

Applying (12), (15) and theorem 2 with $Q^k_{i,j}(t) = L^k_{i,j}(t)$, we obtain from (14):

$$\begin{aligned}
& E\left[V(\tilde{L}(t+1)) - V(L(t)) | L(t)\right] \\
&\leq N(S+1) + 2\sum_{i,j} \lambda_{i,j} L^S_{i,j}(t) - 2\sum_{k=1}^{S}\sum_{i,j} S^k_{i,j}(t) L^k_{i,j}(t) \\
&\leq N(S+1) + 2\sum_{i,j} \lambda_{i,j} L^S_{i,j}(t) - \frac{2}{R}\sum_{k=1}^{S}\sum_{i,j} \lambda_{i,j}(t) L^k_{i,j}(t) \\
&\leq N(S+1) + 2\left[\sum_{i,j} \lambda_{i,j} L^S_{i,j}(t) - \frac{2S}{R}\sum_{i,j} \lambda_{i,j}(t) L^S_{i,j}(t)\right] \\
&\leq N(S+1) + 2\left(1 - \frac{S}{R}\right) \min_{\substack{i,j \\ \lambda_{i,j}>0}} \lambda_{i,j} ||L^S(t)||_1. \qquad (16)
\end{aligned}$$

From (13) and (16), we see

$$\begin{aligned}
E\left[V(\tilde{L}(t+1)) - V(L(t)) | L(t)\right] &\leq N(S+1) + 2\left(1 - \frac{S}{R}\right) \min_{\substack{i,j \\ \lambda_{i,j}>0}} \lambda_{i,j} ||L(t)||_1 \\
&\quad + 2\left(\frac{S}{R} - 1\right) \min_{\substack{i,j \\ \lambda_{i,j}>0}} \lambda_{i,j}\, N(S-1).
\end{aligned}$$

By definition, we note that for a value $b, 0 \leq b \leq S$,

$$L_{i,j}(t+1) - \tilde{L}_{i,j}(t+1) = \begin{Bmatrix} 0, \text{ if } L_{i,j}(t) \geq S, \\ b, \text{ else }. \end{Bmatrix}$$

Thus, $E[V(L(t+1)) - V(\tilde{L}(t+1))|L(t)] \leq N^2S^2$. Now, the theorem follows from theorem 1, because we obtain by (17):

$$E[V(L(t+1)) - V(L(t))|L(t)] \leq -\epsilon||L(t)||_1, \qquad \forall L(t) : ||L(t)|| > B.$$

For the analysis of the $MM-OCF$ algorithm, we denote by $C^k_{i,j}(t)$ the head of line cell of $VOQ_{i,j}$ at the beginning of the k-th internal timeslot of the t-th external timeslot. We define the interarrival vector $\tau^k(t) = (\tau^k_{1,1}(t), .., \tau^k_{N,N}(t))$, where $\tau^k_{i,j}(t)$ is the interarrival time between $C^k_{i,j}(t)$ and the cell behind it in line. $W^k_{i,j}(t)$ denotes the actual waiting time of $C^k_{i,j}(t)$. We set $W_{i,j}(t) = W^1_{i,j}(t)$, and describe the evolution of $W^k_{i,j}(t)$ between consecutive internal timeslots:

$$W^k_{i,j}(t) = [W^{k-1}_{i,j}(t) - S^{k-1}_{i,j}(t)\tau^{k-1}_{i,j}(t)]^+, \tag{17}$$

$$W^1_{i,j}(t) = [W^S_{i,j}(t-1) - S^S_{i,j}(t-1)\tau^S_{i,j}(t-1) + 1]^+, \tag{18}$$

$\forall k$, $2 \leq k \leq S$. Combining the last two equations, we obtain:

$$W_{i,j}(t+1) = \left[W_{i,j}(t) - \sum_{k=1}^{S} S^k_{i,j}(t)\tau_{i,j}(t) + 1 \right]^+. \tag{19}$$

The approximate next-state vectors are defined $\forall k$, $2 \leq k \leq S$ as follows:

$$W^k_{i,j}(t) = W^{k-1}_{i,j}(t) - S^{k-1}_{i,j}(t)\tau^{k-1}_{i,j}(t), \tag{20}$$

$$W^1_{i,j}(t) = W^S_{i,j}(t-1) - S^S_{i,j}(t-1)\tau^S_{i,j}(t-1) + 1, \tag{21}$$

$$\tilde{W}_{i,j}(t+1) = W^k_{i,j}(t) - \sum_{k=1}^{S} S^k_{i,j}(t)\tau^k_{i,j}(t) + 1.$$

We argue as in (12) and (13) and use the relation $E[\tau^k_{i,j}(t)] = \lambda^{-1}_{i,j}$ to obtain:

$$W^S_{i,j}(t) \leq W^k_{i,j}(t) \ \forall k, \ 2 \leq k \leq S, \tag{22}$$

$$E\left[||W^S(t)||_1\right] \geq E\left[||W(t)||_1\right] - (S-1)N \min_{\substack{i,j \\ \lambda_{i,j}>0}} \lambda^{-1}_{i,j}. \tag{23}$$

Theorem 4: *The $MM-OCF$ algorithm with a speedup $S > R$ is stable for all admissible i.i.d. arrival processes.*
Proof (Sketch): We follow the proof of theorem 3. As in [6], we use the Lyapunov function $V(W(t))$ defined as $V(W(t)) = \sum_{i,j} W^2_{i,j}(t)\lambda_{i,j}$. Applying (20) and (21) instead of (9) and (10), we obtain

$$E\left[V(\tilde{W}(t+1)) - V(W(t))\right] = 2\sum_{i,j} \lambda_{i,j} W^S_{i,j}(t) - 2\sum_{k=1}^{S} S^k_{i,j}(t)W^k_{i,j}(t)$$

$$+\sum_{i,j}\left[\lambda_{i,j}-2\sum_{k=1}^{S}S_{i,j}^{k}(t)+\sum_{k=1}^{S}\frac{S_{i,j}^{k}(t)}{\lambda_{i,j}}\right].$$

The sequel of the proof follows the proof of theorem 3. Instead of the relations (12) and (13), we use the relations (22) and (23). Instead of (17), we use:

$$W_{i,j}(t+1)=\begin{Bmatrix}\tilde{W}_{i,j}(t+1), \text{ if } W_{i,j}(t+1)\geq 0,\\ 0, \qquad\qquad \text{else.}\end{Bmatrix} \qquad \square$$

5 Bounds on average delay

In this section, we derive bounds on the average delay experienced by cells at the VOQs. We first consider the $MM-LQF$ algorithm and prove the following bound on the average sum of the length of all VOQs.

Theorem 5: *Under the assumption of i.i.d. admissible traffic, for the $MM-LQF$ algorithm with a speedup $S>R$, the expected sum of all the queue lengths $E[||L(t)||_1]$ is bounded as follows:*

$$E[||L(t)||_1]\leq\frac{\sum\limits_{i,j}\lambda_{i,j}}{\left(\frac{S}{R}-1\right)\min\limits_{\substack{i,j\\ \lambda_{i,j}>0}}\lambda_{i,j}}+N. \tag{24}$$

Proof: Following an argument in [7], we revisit the proof of theorem 3. We assume that $L(t+1)=\tilde{L}(t+1)$. An analysis of the proof below shows that the exact state vector would incur an additional term of $N^2(S+1)$ to the delay bound. The approximate state vector will give a bound C. We will see that in general $C\geq N^2(S+1)$, and hence we will not consider the term $N^2(S+1)$ in this section. Arguing as in (14) and (16), without using the estimate (15), we obtain:

$$V(L(t+1))-V(L(t))\leq 2\left(1-\frac{S}{R}\right)\min_{\substack{i,j\\ \lambda_{i,j}>0}}\lambda_{i,j}||L^S(t)||_1+\sum_{i,j}A_{i,j}^2(t+1)$$
$$-2\sum_{i,j}A_{i,j}(t+1)S_{i,j}^{S}(t)+\sum_{k=1}^{S}\sum_{i,j}\left(S_{i,j}^{k}(t)\right)^2. \tag{25}$$

Following an argument from [7], we see that for large enough t.

$$E[A_{i,j}(t)]=\lambda_{i,j},\quad E[A_{i,j}^2(t)]=\lambda_{i,j}, \tag{26}$$

As the $MM-LQF$ algorithm is stable, the switch state becomes a discrete time Markov chain with a stationary distribution. Thus, for large enough t,

$$E\left[\sum_{k=1}^{S}S_{i,j}^{k}(t)\right]=E\left[\sum_{k=1}^{S}\left(S_{i,j}^{k}(t)\right)^2\right]=E[A_{i,j}(t)]=\lambda_{i,j}. \tag{27}$$

In summary, from (25)-(27) we obtain:

$$E[V(L(t+1)) - V(L(t))|L(t)] \leq 2\sum_{i,j} \lambda_{i,j} + 2 \min_{\substack{i,j \\ \lambda_{i,j}>0}} \lambda_{i,j} \left(1 - \frac{S}{R}\right) E[||L(t)||_1].$$

We use the last equation to obtain:

$$\begin{aligned} E[V(L(t+1))] &= E[V(L(t+1)) - V(L(t)) + V(L(t))] \\ &\leq 2\sum_{i,j} \lambda_{i,j} + 2 \min_{\substack{i,j \\ \lambda_{i,j}>0}} \lambda_{i,j} \left(1 - \frac{S}{R}\right) E[||L^S(t)||_1] + E[V(L(t))]. \end{aligned}$$

Summing over $t = 0$ to $t = T - 1$, we get

$$E[V(L(T))] \leq 2T\sum_{i,j} \lambda_{i,j} + 2 \min_{\substack{i,j \\ \lambda_{i,j}>0}} \lambda_{i,j} \left(1 - \frac{S}{R}\right) \sum_{t=0}^{T-1} E||L^S(t)||_1 + E[V(L(0))].$$

We see from (3) and (10) that $||L_{i,j}(t+1)||_1 \leq ||L^S_{i,j}(t)||_1 + N$. Thus, noting that $V(\cdot) \geq 0$ and assuming $E[V(L(0))] = 0$, we derive from the last equation:

$$\frac{1}{T}\sum_{t=0}^{T} E[||L(t+1)||_1] \leq \frac{\sum_{i,j} \lambda_{i,j}}{\left(\frac{S}{R} - 1\right) \min_{\substack{i,j \\ \lambda_{i,j}>0}} \lambda_{i,j}} + N. \tag{28}$$

As we assume an i.i.d. arrival process, the switch state is a discrete, irreducible aperiodic Markov chain. Thus, it is ergodic, and the left hand side of (28) converges to the expected value of $||L(T+1)||_1$ in the equilibrium state, i.e., $\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} E[||L(t+1)||_1] = \lim_{T\to\infty} E[||L(T+1)||_1]$. Inserting this relation in (28), (24) follows.

For the specific case of uniform arrival traffic for all VOQs, i.e., $\lambda_{i,j} = \lambda \,\forall\, i, j$, $1 \leq i, j \leq N$, there holds $E[L(t)]/N^2 = E[L_{i,j}(t)]$. Using Little's law, we derive from theorem 5 that the expected delay $E[T]$ is bounded as:

$$E[T] \leq \frac{1}{\left(\frac{S}{R} - 1\right)\lambda} + \frac{1}{N\lambda}. \tag{29}$$

For the $MM - OCF$ algorithm, the following theorem holds:

Theorem 6: *Under the assumption of i.i.d. admissible traffic, for the $MM - OCF$ algorithm with a speedup $S > R$, the expected sum of all the delays in all virtual output queues $E[W(t)]$ is bounded as follows:*

$$E[||W(t)||_1] \leq \frac{N^2 + \sum_{i,j} \lambda_{i,j}}{2\left(\frac{S}{R} - 1\right) \min_{\substack{i,j \\ \lambda_{i,j}>0}} \lambda_{i,j}} + N.$$

Proof: The proof follows the proof of theorem 5. □

Assuming uniform traffic, a bound on the average delay is obtained by dividing the right side of theorem 6 by N^2. This differs from the derivation of the delay bound (29) from theorem 5 which was performed using Little's law.

6 Absolute delay bound for the $MM - OCF$ algorithm

In this section, we use techniques first introduced in [2] to establish a bound on the absolute delay for the $MM - OCF$ algorithm.

Theorem 7 *There exists a constant F, such that under the assumption of i.i.d. admissible traffic, for the $MM - OCF$ algorithm with a speedup $S \geq R$, the absolute maximum delay D a cell can experience at an input queue is bounded as $D \leq \frac{(2N-1)F+1}{S-R}$.*

Proof: We follow the proof of theorem 5 in [2]. Whereas in [2] the arrival rate of all cells that compete for resources with a given cell was bounded by 2, we bound it by R. The leaky bucket conditions and the constant F needed for the proof in [2] can be derived from (1).

7 Conclusions

This paper presents a new approach to establish results on stability and delay bounds for maximal weight matching algorithms. The proof is based on a new mathematical model of the dynamics of a maximal weight matching algorithm and on the theory of the Lyapunov function. This approach allows for the first time to give delay bounds for maximal weight matching algorithms with a speedup of less than two. Furthermore, an absolute delay bound for a maximal weight matching algorithm with a speedup of less than two is proved.

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