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A note on sums of five almost equal prime squares

By

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Abstract. Let N be any sufficiently large positive integer satisfying the congruence condition $N \equiv 5 (\bmod 24)$. It is shown that there exists a $\delta > 0$ such that N can be written as

$$\begin{cases} N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2, \\ \left| p_j - \sqrt{\frac{N}{5}} \right| \leqq U, \quad j = 1, 2, 3, 4, 5, \end{cases}$$

where the p_i are prime numbers and U is chosen as $U = N^{\frac{1}{2} - \delta}$.

1. Introduction and statement of results. One of Hua's outstanding contributions to prime number theory was to prove that every sufficiently large integer $N \equiv 5 (\bmod 24)$ can be written as the sum of five prime squares ([3]). Recently, Liu and Zhan ([6]) were able to sharpen this result in the following way:

Theorem 1. *Assume the Great Riemann Hypothesis. Denote by $R(N, U)$ the number of solutions of the Diophantine equation with prime variables*

$$\begin{cases} N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2, \\ \left| p_j - \sqrt{\frac{N}{5}} \right| \leqq U, \quad j = 1, 2, 3, 4, 5. \end{cases}$$

Then for $U = N^{\frac{9}{20} + \varepsilon}$, we have

$$R(N, U) = \frac{460\sqrt{5}}{3} \sigma(N) \frac{U^4}{N^{\frac{1}{2}} \log^5 N} (1 + o(1)),$$

where

$$\sigma(N) = \sum_{q=1}^{\infty} \frac{1}{\varphi^5(n)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^5(a, q) e\left(-\frac{aN}{q} \right)$$

with

$$C(a, q) = \sum_{\substack{a=1 \\ (h,q)=1}}^{q} e\left(\frac{ah^2}{q} \right).$$

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Here $\sigma(N)$ *is the so-called singular series, which is convergent and satisfies* $\sigma(N) > c > 0$ *for* $N \equiv 5 \pmod{24}$.

The proof uses the circle method. The unit interval is in the usual way split into *major arcs* and *minor arcs*. The contribution derived from the *minor arcs* is estimated by the following theorem which is also proved in [6]:

Theorem 2. *Let* $\varepsilon > 0$ *be arbitrary,* $1 \leqq y \leqq x$ *and*

$$S_2(x, y; \alpha) = \sum_{x < n \leqq x+y} \Lambda(n) e(n^2 \alpha).$$

Then

$$S_2(x, y; \alpha) \ll y^{1+\varepsilon} \left(\frac{1}{q} + \frac{x^{\frac{1}{2}}}{y} + \frac{x^{\frac{4}{3}}}{y^2} + \frac{qx}{y^3} \right)^{\frac{1}{4}} \tag{1.1}$$

holds for $\alpha = \dfrac{a}{q} + \lambda$, $(a, q) = 1$ *satisfying* $1 \leqq q \leqq xy$, $|\lambda| \leqq \dfrac{1}{q^2}$.

We will show in this paper that Theorem 1 holds in a weaker form without assuming any hypothesis on the distributions of the zeros of the L-functions. More precisely, we will prove:

Theorem. *There exists a* $\delta > 0$ *such that every sufficiently large number* $N \equiv 5 \pmod{24}$ *can be written as*

$$\begin{cases} N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2, \\ \left| p_j - \sqrt{\dfrac{N}{5}} \right| \leqq U, \quad j = 1, 2, 3, 4, 5, \end{cases}$$

with U *chosen as*

$$U = N^{\frac{1}{2} - \delta}. \tag{1.2}$$

We will adopt a method developed by Liu and Tsang ([4], [5]) to our problem in order to calculate the contribution of the *major arcs*. Because we follow very closely the work of Liu and Tsang we will often not give all the details of the proof, but refer to the corresponding arguments in [4] and [5]. The *minor arcs* will be treated by Theorem 2 as in [6] .

2. Notation and structure of the proof. The most part of our notations will be chosen similar to the notations in [5]. Throughout this paper p always denotes a prime number; $c_1, c_2, \ldots$ are effective positive constants and δ denotes a small positive number, which will be specified later. U is defined by (1.2) and further let

$$L = \log N, \quad P = N^{\delta_1}, \quad T = P^{1/\sqrt{\delta_1}}, \quad Q = NT^{-1/4},$$

where $\delta_1 = 104\,\delta$. It is a well known fact (see [1]) that there is at most one primitive character to a modulus $q \leqq T$ for which the corresponding L-function has a zero in the region

$$\sigma < 1 - \eta(T), \quad |t| \leqq T, \quad \text{where} \quad \eta(T) = \frac{c_1}{\log T}, \tag{2.1}$$

for a small constant c_1. If there is such an exceptional character, it is real and we denote it by $\tilde{\chi}$. The corresponding exceptional zero is real, simple and unique, and we denote it by $\tilde{\beta}$. If $\tilde{\chi}$ exists, the zero-free region in (2.1) is widened to (see [2])

$$\eta(T) = \frac{c_2}{\log T} \log \left(\frac{ec_1}{(1-\tilde{\beta})\log T} \right). \tag{2.2}$$

It is further known that for the exceptional modul $\tilde{r}$ the estimates

$$\frac{c_3}{\tilde{r}^{1/2} \log^2 \tilde{r}} \leqq 1 - \tilde{\beta} \leqq \frac{c_1}{\log T} \tag{2.3}$$

hold. For any $x > N^{1/4}$ and any $\chi \bmod q$ with $q \leqq T$ we define:

$$S_\chi(x, T) = {\sum_{|\gamma| \leqq T}}' x^{\beta - 1},$$

where ${\sum_{|\gamma| \leqq T}}'$ denotes the summation over all zeros $= \beta + i\gamma$ of $L(s, \chi)$ lying inside the region: $|\gamma| \leqq T, \frac{1}{2} \leqq \beta \leqq 1 - \eta(T)$ and $\eta(T)$ is defined in (2.2) or (2.1) according as $\tilde{\beta}$ exists or not. Let

$$\Omega(T) = \begin{cases} (1-\tilde{\beta})\log T, & \text{if } \tilde{\beta} \text{ exists}, \\ 1, & \text{otherwise}. \end{cases} \tag{2.4}$$

Using these results it can be shown by applying Gallagher's density estimate ([2]) that the following lemma, which is shown in the same way as Lemma 2.1 in [4], is true.

Lemma 2.1. *If $x \geqq N^{1/4}$ there exists an absolute constant c_4 such that for a sufficiently small δ_1*

$$\sum_{q \leqq T} {\sum_{\chi \bmod q}}^* S_\chi(x, T) \ll \Omega^5(T) \exp(-c_4/\delta_1),$$

where ${\sum_{\chi \bmod q}}^*$ denotes the summation over all primitive characters $\chi \pmod q$.

Further for any real λ we set $e(\lambda) = e^{2\pi i \lambda}$ and

$$N_1 = \sqrt{\frac{N}{5}} - U, \quad N_2 = \sqrt{\frac{N}{5}} + U,$$

which we use to define

$$S(\alpha) = \sum_{N_1 < n \leqq N_2} \Lambda(n) e(n^2 \alpha), \quad S_\chi(\alpha) = \sum_{N_1 < n \leqq N_2} \Lambda(n) \chi(n) e(n^2 \alpha),$$

for every character $\chi \pmod q$ with $q \leqq T$.

$$I(\alpha) = \int_{N_1}^{N_2} e(x^2 \alpha)\, dx, \quad \tilde{I}(\alpha) = \int_{N_1}^{N_2} x^{\tilde{\beta}-1} e(x^2 \alpha)\, dx,$$

and

$$I_\chi(\alpha) = \int_{N_1}^{N_2} e(x^2 \alpha) {\sum_{|\gamma| \leqq T}}' x^{\rho - 1}\, d.$$

For any character χ mod q let

$$C_\chi(m) = \sum_{l=1}^{q} \chi(l) e\left(\frac{ml^2}{q}\right), \quad C_q(m) = C_{\chi_0}(m).$$

We write $\sum_{a=1}^{q}{}^* = \sum_{\substack{a=1\\(a,q)=1}}^{q}$, recall $Q = NT^{-1/4}$ and define the *major arcs* and *minor arcs* as follows:

$$M = \sum_{q \leqq P} \sum_{a=1}^{q}{}^* I(a,q), \; I(a,q) = \left[\frac{a}{q} - \frac{1}{Qq}, \frac{a}{q} + \frac{1}{Qq}\right],$$

$$m = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus M.$$

The *major arcs* are obviously disjoint subintervals of $\left[\frac{1}{Q}, 1 + \frac{1}{Q}\right]$. Writing

(2.5) $$I(n) = \sum_{\substack{N_1 < n_1, \dots, n_5 \leqq N_2\\ n_1^2 + \dots + n_5^2 = N}} \Lambda(n_1) \dots \Lambda(n_5),$$

we obtain

(2.6) $$I(N) = \int_{\frac{1}{Q}}^{1+\frac{1}{Q}} e(-n\alpha) S^5(\alpha)\, d\alpha = \left(\int_M + \int_m\right) e(-n\alpha) S^5(\alpha)\, d\alpha =: I_1(N) + I_2(N).$$

We will first treat the integral over the *major arcs*.

3. Simplification of $I_1(N)$. For any α in $I(a,q)$ we have $\alpha = \frac{a}{q} + \eta$ with $|\eta| \leqq \frac{1}{qQ}$. In a well known way we obtain

(3.1) $$S(\alpha) = \phi^{-1}(q) \sum_{\chi \bmod q} C_{\bar{\chi}}(a) S_\chi(\eta).$$

Following the arguments in [4] we will now give four lemmas which we will use to simplify the contribution of the *major arcs*. Their proofs will not always be given completely because some of them can be shown in exactly the same way as Lemma 3.1. to 3.4. in [4].

Lemma 3.1. *For any real α and any $\chi \bmod q$ with $q \leqq T$, we obtain*

$$S_\chi(\eta) = \delta_{\chi_0} I(\eta) - \delta_{\tilde{\chi}} \tilde{I}(\eta) - I_\chi(\eta) + O((1 + |\eta|N) N^{1/2} L^2 T^{-1}),$$

where

$$\delta_{\chi_0} = \begin{Bmatrix} 1, & \text{if } \chi = \chi_0 \,(\mathrm{mod}\, q) \\ 0, & \text{otherwise,} \end{Bmatrix}, \quad \delta_{\tilde{\chi}} = \begin{Bmatrix} 1, & \text{if } \chi = \tilde{\chi}\chi_0 \,(\mathrm{mod}\, q) \\ 0, & \text{otherwise.} \end{Bmatrix}$$

Proof. We note that for $2 \leqq T \leqq x$ the identity (see [1], p. 109 and p. 120.)

(3.2) $$\sum_{n \leqq x} \chi(n) \Lambda(n) = \delta_{\chi_0} x - \delta_{\tilde{\chi}} \frac{x^{\tilde{\beta}}}{\tilde{\beta}} - \sum_{|Im\rho| \leqq T}{}' \frac{x^\rho}{\rho} + R(x,q)$$

is valid with $R(x,q) \ll \frac{xL^2}{T} + L$ and the summation is running over all zeros of $L(s,\chi)$ with $0 \leqq Re(\rho) \leqq 1, |Im(\rho)| \leqq T$ and the possible Siegel – zero is excluded. Then Lemma 3.1. follows by using partial summation if we note that

$$\int_{N_1}^{N_2} e(x^2\eta)\, d(R(x,q) \ll N^{1/2}L^2T^{-1} + \int_{N_1}^{N_2} |R(x,q)| \left|\frac{d}{dx} e(x^2\eta)\right|$$
$$\ll (1+|\eta|N)N^{1/2}L^2T^{-1}.$$

Lemma 3.2. *Let $\rho = \beta + i\gamma$, $1/2 \leqq \beta \leqq 1$. Then for any real η it is known that*

$$\int_{N_1}^{N_2} e(x^2\eta)x^{\rho-1}dx \ll \left\{ \begin{array}{ll} \min\left(N_2^\beta, |\eta|^{-\frac{\beta+1}{2}}N_1^{-1}\right), & \text{if } \gamma = 0, \\ N_2^\beta|\gamma|^{-1}, & \text{if } |\eta| \leqq \dfrac{|\gamma|}{8\pi N_2^2}, \\ N_2^2N_1^{\beta-2}|\gamma|^{-1/2}, & \text{if } \dfrac{|\gamma|}{8\pi N_2^2} \leqq |\eta| \leqq \dfrac{|\gamma|}{2\pi N_1^2}, \\ N_1^{\beta-2}|\eta|^{-1}, & \text{if } \dfrac{|\gamma|}{2\pi N_1^2} < |\eta|. \end{array} \right\}$$

The proof of this lemma is literally the same as the one of Lemma 3.2 in [4].

Lemma 3.3. *For any real η we obtain*

$$I(\eta) \ll \min\left(N_2, |\eta|^{-1}N_1^{-1}\right), \quad \tilde{I}(\eta) \ll \min\left(N_2^{\tilde{\beta}}, |\eta|^{-\frac{\tilde{\beta}+1}{2}}N_1^{-1}\right),$$

$$I_\chi(\eta) \ll \left\{ \begin{array}{ll} N_2, & \text{for any real } \eta, \\ N_2^2N_1^{-2}(|\eta|)^{-1/2}, & \text{for } N_2^{-2} < |\eta| \leqq \dfrac{T}{2\pi N_1^2}, \\ |\eta|^{-1}N_1^{-1}, & \text{for } \dfrac{T}{2\pi N_1^2} < |\eta|. \end{array} \right\}$$

Using lemma 3.2 this lemma is proved in exactly the same way as Lemma 3.3 in [4].

Lemma 3.4.

$$\int_{-\infty}^{\infty} |I(\eta)|^4\, d\eta \ll \frac{N_2^6}{N_1^4}, \quad \int_{-\infty}^{\infty} |\tilde{I}(\eta)|^4\, d\eta \ll \frac{N_2^{3(\tilde{\beta}+1)}}{N_1^4},$$

$$\int_{-\infty}^{\infty} |I_\chi(\eta)|^4\, d\eta \ll \frac{N_2^{10}}{N_1^8}.$$

Proof. The first estimate follows from Lemma 3.3, if we split up the integral in the following way:

$$\int_{-\infty}^{\infty} |I(\eta)|^4\, d\eta \ll \int_{|\eta| \leqq N_2^{-2}} N_2^4\, d\eta + \int_{N_2^{-2}<|\eta|} |\eta|^{-4}N_1^{-4}d\eta \ll \frac{N_2^6}{N_1^4}.$$

The second estimate is proved in the same way whereas for the proof of the third estimate we split the integral in the following way:

$$\int_{-\infty}^{\infty} |I_\chi(\eta)|^4\,\delta\eta \ll \int_{|\eta| \leqq N_2^{-2}} N_2^4\,d\eta + \int_{N_2^{-2}<|\eta| \leqq \frac{T}{2\pi N_1^2}} N_2^8 N_1^{-8}(|\eta|)^{-2}\,d\eta$$
$$+ \int_{\frac{T}{2\pi N_1^2}<|\eta|} |\eta|^{-4} N_1^{-4}\,d\eta \ll N_2^2 + \frac{N_2^{10}}{N_1^8} + \frac{N_1^2}{T^3} \ll \frac{N_2^{10}}{N_1^8}.$$

We now simplify $I_1(N)$ in the same way as it is done in [4]. Set

$$G(a,q,\eta) = \sum_{\chi \bmod q} C_{\bar{\chi}}(a) I_\chi(\eta)$$

and

$$H(a,q,\eta) = C_q(a) I(\eta) - \delta_q C_{\tilde{\chi}\chi_0}(a)\tilde{I}(\eta) - G(a,q,\eta),$$

where

$$\delta_q = \left\{\begin{matrix} 1, & \text{if } \tilde{r}|q, \\ 0, & \text{otherwise.}\end{matrix}\right\}$$

For any $\alpha = \frac{a}{q} + \eta \in I(a,q)$ we obtain by applying Lemma 3.1 to (3.1)

$$S(\alpha) = \phi^{-1}(q)\left(H(a,q,\eta) + O\left(\sum_{\chi \bmod q} (1+|\eta|N)|C_{\bar{\chi}}(a)|N^{1/2}L^2T^{-1}\right)\right).$$

From the definition of the *major arcs* we see that $|\eta|N \leqq T^{1/4}$ and trivially we find that

$$\left|\sum_{\chi \bmod q} C_{\bar{\chi}}(a)\right| \leqq \phi^2(q).$$

So the O-term above is $\ll \phi(q)N^{1/2}L^2T^{-3/4}$. Together with the definition of $I_1(N)$ we obtain

$$I_1(N) = \sum_{q \leqq P} \phi^{-5}(q) \sum_{a=1}^{q}{}^{*}\, e\left(-\frac{a}{q}N\right)$$
$$\cdot \int_{1/qQ}^{1/qQ} e(-\eta N)\big(H(a,q,\eta) + O(\phi(q)N^{1/2}L^2T^{-3/4})\big)^5\,d\eta.$$

It is easily deduced from Lemma 3.3 that $H(a,q,\eta) \ll \phi^2(q)N^{1/2}$. Using this relation we see that the grand error term in the last expression for $I_1(N)$ may be estimated by

$$\ll \sum_{q \leqq P} \phi^{-5}(q) \sum_{a=1}^{q}{}^{*} \int_{1/qQ}^{1/qQ} (\phi^2(q)N^{1/2})^4 \phi(q)N^{1/2}L^2T^{-3/4}d\eta$$
$$\ll N^{3/2}P^{-2} \leqq U^4N^{-1/2}P^{-1}$$

for a sufficiently small δ_1 and because of $\delta_1 = 104\,\delta$. Hence we reach

$$I_1(N) = \sum_{q \leqq P} \phi^{-5}(q) \sum_{a=1}^{q}{}^{*}\, e\left(-\frac{a}{q}N\right) \int_{1/qQ}^{1/qQ} e(-\eta N)H^5(a,q,\eta)\,d\eta + O(U^4N^{-1/2}P^{-1}). \tag{3.3}$$

The next step will be to extend the range of integration in (3.3) to $(-\infty, \infty)$. The product $H^5(a,q,\eta)$ is a sum of $(\phi(q)+2)^5$ terms of the form $\prod_{j=1}^{5} E_j$, where each E_j is $C_q(a)I(\eta), -\delta_q C_{\bar{\chi}\chi_0}(a)\tilde{I}(\eta)$ or $-C_{\bar{\chi}}(a)I_\chi(\eta)$. We note that for $|\eta| \geqq (qQ)^{-1}$ among the estimates for $I(\eta), \tilde{I}(\eta)$ and $I_\chi(\eta)$ in Lemma 3.3 the weakest one is the estimate in the middle range for $I_\chi(\eta)$. So we obtain

$$\int\limits_{|\eta|>(qQ)^{-1}} \prod_{j=1}^{5} E_j \, d\eta \ll \phi(q)(qQ)^{1/2} \int\limits_{-\infty}^{\infty} |E_1E_2E_3E_4| \, d\eta.$$

Because of Lemma 3.4 this is $\ll \phi^5(q)N(qQ)^{1/2}$. Thus extending the integration to $(-\infty, \infty)$, the total error induced is

$$\ll \sum_{q \leqq P} \phi^{-5}(q)\phi(q)(\phi(q)+2)^2\phi^5(q)q^{1/2}N^{3/2}T^{-1/8} \ll N^{3/2}P^{-2} \leqq U^4N^{-1/2}P^{-1}$$

for a sufficiently small δ_1 and because of $\delta_1 = 104\delta$. So (3.3) can now be written as

$$I_1(N) = \sum_{q \leqq P} \phi^{-5}(q) \sum_{a=1}^{q}{}^{*} e\left(-\frac{a}{q}N\right) \int\limits_{-\infty}^{\infty} e(-\eta N)H^5(a,q,\eta)\, d\eta + O(U^4N^{-1/2}P^{-1}). \tag{3.4}$$

4. Final treatment of the major arcs. The following treatment of the *major arcs* is nearly identical with the procedure in [5]. For the treatment of the singular series we can completely refer to [5]. We recall the definitions $N_1 = \sqrt{\frac{N}{5}} - U$, $N_2 = \sqrt{\frac{N}{5}} + U$. We use the following lemma for the calculation of the contribution of the major arcs:

Lemma 4.1. *For any complex numbers ρ_i with $0 < Re(\rho_j) \leqq 1, j = 1, ..., 5$, it is known that*

$$\int\limits_{-\infty}^{\infty} e(-N\eta) \prod_{j=1}^{5} \left(\int\limits_{N_1}^{N_2} x^{\rho_j-1} e(\eta x^2)\, dx \right) d\eta = 2^{-5}N_2^3 \int\limits_{\mathscr{D}} \prod_{j=1}^{5} (N_2^2 x_j)^{(\rho_j-1)/2} x_j^{-1/2}\, dx_1 \dots dx_4, \tag{4.1}$$

where

$$x_5 = NN_2^{-2} - \sum_{j=1}^{4} x_j \tag{4.2}$$

and

$$\mathscr{D} = \{(x_1, \dots, x_4) : (N_1/N_2)^2 \leqq x_1, \dots, x_5 \leqq 1\}. \tag{4.3}$$

Furthermore the lower estimate

$$\int\limits_{\mathscr{D}} \left(\prod_{j=1}^{5} x_j^{-1/2} \right) dx_1 \dots dx_4 \gg U^4 N^{-2} \tag{4.4}$$

holds.

Proof. (4.1) is shown in exactly the same way as (3.15) in [5]. For the proof of (4.4) we note that because of (4.2) the condition for x_5 in (4.3) is equivalent to

$$\frac{N}{N_2^2} - 1 \leqq \sum_{j=1}^{4} x_j \leqq \frac{N - N_1^2}{N_2^2}. \tag{4.5}$$

We now define the region $\mathscr{D}_1$ by

$$\mathscr{D}_1 = \left\{ (x_1, \ldots, x_4) : \ (N_1/N_2)^2 \leqq x_1, \ldots, x_4 \leqq \frac{N - N_1^2}{4N_2^2} \right\}$$

and show that it lies in $\mathscr{D}$. Taking into account that $\dfrac{N}{N_2^2} - 1 < 0$ we see from (4.3) and (4.5) that the lower bounds of $\mathscr{D}_1$ are equal to those of $\mathscr{D}$. This together with the relation

$$\frac{N - N_1^2}{4N_2^2} = \frac{\frac{4}{5}N + 2\sqrt{\frac{N}{5}}U - U^2}{\frac{4}{5}N + 8\sqrt{\frac{N}{5}}U + 4U^2} < 1$$

shows that $\mathscr{D}_1$ lies in $\mathscr{D}$. Using $x_j^{-1/2} \geqq 1$ we find that

$$\int\limits_{\mathscr{D}} \left(\prod_{j=1}^{5} x_j^{-1/2} \right) dx_1 \ldots dx_4 \geqq \int\limits_{\mathscr{D}_1} \left(\prod_{j=1}^{4} x_j^{-1/2} \right) dx_1 \ldots dx_4 \geqq \left(\frac{N - N_1^2}{4N_2^2} - \left(\frac{N_1}{N_2} \right)^2 \right)^4$$

$$= \left(\frac{N - 5N_1^2}{4N_2^2} \right)^4 = \left(\frac{10\sqrt{\frac{N}{5}}U - 5U^2}{4N_2^2} \right)^4 \gg U^4 N^{-2},$$

which proves (4.4).

We know from the definition of $H(a, q, \eta)$ that $H^5(a, q, \eta)$ is a sum of 3^5 terms which can be divided into three groups:

T_1: the term $(C_q(a) I(\eta))^5$,
T_2: the 211 terms each of which has at least one $G(a, q, \eta)$ as factor,
T_3: the remaining 31 terms.

We further write for $i = 1, 2, 3$

$$M_i = \sum_{q \leqq P} \phi^{-5}(q) \sum_{a=1}^{q}{}^{*} e\left(\frac{-Na}{q} \right) \int\limits_{-\infty}^{\infty} e(-N\eta) \{\text{sum of the terms in } T_i\} \, d\eta,$$

from which we deduce by using (3.4)

$$I_1(N) = M_1 + M_2 + M_3 + O(U^4 N^{-1/2} P^{-1}). \tag{4.6}$$

We also define

$$\mathscr{P}_0 = \frac{N_2^3}{2^5} \int\limits_{\mathscr{D}} \left(\prod_{j=1}^{5} x_j^{-1/2} \right) dx_1 \ldots dx_4, \tag{4.7}$$

$$\sum_{(q)} \chi_1(n_1) \ldots \chi_5(n_5) = \sum_{\substack{1 \leqq n_1, \ldots, n_5 \leqq q,\, (n_j, q) = 1 \\ n_1^2 + \ldots + n_5^2 \equiv N \, (\mathrm{mod}\, q)}} \chi_1(n_1) \ldots \chi_5(n_5),$$

and

$$s(p) = \left\{ \begin{array}{ll} \phi^{-5}(2^3) 2^3 \sum\limits_{(2^3)} 1 & \text{for } p = 2, \\ \phi^{-5}(p) p \sum\limits_{(p)} 1 & \text{for } p \geqq 3. \end{array} \right\}.$$

Without further mentioning it we will make use of the fact that $\prod_p s(p) \gg 1$. Finally we know from (4.10) in [5] that

$$\prod_{p|\tilde{r}} s(p) = \sigma\tilde{r}\phi^{-5}(\sigma\tilde{r}) \sum_{(\sigma\tilde{r})} 1 \tag{4.8}$$

holds, where $\sigma = 1, 4$ and 2 for $2\nmid\tilde{r}$, $2 \parallel \tilde{r}$ and $4|\tilde{r}$ respectively. We will now give estimates for the respective contribution of the M_i to $I_1(N)$ from which we can easily calculate the contribution of the *major arcs*. We first have

$$M_1 = \mathscr{P}_0 \prod_p s(p) \;\; + O(N^{3/2}P^{-1}\log^{60} P), \tag{4.9}$$

the proof of which is literally the same as the one of Lemma 4.1 in [5]. The next estimates are given by

$$M_3 \ll N_2^3\tilde{r}^{-1}\log P \tag{4.10}$$

and

$$M_1 + M_3 \gg \Omega^5\mathscr{P}_0 \prod_p s(p) \;\; + O(N^{3/2}P^{-1}\log^{60} P). \tag{4.11}$$

(4.10) corresponds to Lemma 4.2 b) in [5]. If $\tilde{\beta}$ does not exist the term M_3 does not appear and (4.11) follows from (4.9) and the definition of Ω. In the other case we follow the proof of Lemma 4.3 in [5] and derive in exactly the same way

$$\begin{aligned} M_1 + M_3 = {} & \sigma\tilde{r}\phi^{-5}(\sigma\tilde{r}) \prod_{(p,\tilde{r})=1} s(p) \frac{N_2^3}{2^5} \sum_{(\sigma\tilde{r})} \int_{\mathscr{D}} \left(\prod_{j=1}^{5} x_j^{-1/2} \right) \\ & \times \left(\prod_{j=1}^{5} (1 - \tilde{\chi}(n_j)(N_2^2 x_j)^{(\tilde{\beta}-1)/2}) \right) dx_1 \ldots dx_4 + O(N^{3/2}P^{-1}\log^{60} P). \end{aligned} \tag{4.12}$$

Taking into account that for $x_j \in \mathscr{D}$ there is $x_j \geqq \dfrac{N_1^2}{N_2^2}$ we obtain

$$\left(\prod_{j=1}^{5} (1 - \tilde{\chi}(n_j)(N_2^2 x_j)^{(\tilde{\beta}-1)/2}) \right) \geqq \prod_{j=1}^{5} (1 - N_1^{\tilde{\beta}-1}).$$

Using the mean value theorem of differential calculus we further obtain

$$1 - N_1^{\tilde{\beta}-1} \gg (1 - \tilde{\beta})\log N_1 \gg (1 - \tilde{\beta})\log T = \Omega.$$

Thus we can conclude from (4.12)

$$M_1 + M_3 \gg \Omega^5 \sigma\tilde{r}\phi^{-5}(\sigma\tilde{r}) \prod_{(p,\tilde{r})=1} s(p) \frac{N_2^3}{2^5} \sum_{(\sigma\tilde{r})} \int_{\mathscr{D}} \left(\prod_{j=1}^{5} x_j^{-1/2} \right) + O(N^{3/2}P^{-1}\log^{60} P), \tag{4.13}$$

which together with (4.7) and (4.8) proves (4.11). The contribution of M_2 is estimated in the same way as the corresponding term in [5]. Thus we reach

$$M_2 \ll \Omega^5 \exp(-c/\sqrt{\delta_1}) \mathscr{P}_0 \prod_p s(p) \;\; + O(N^{3/2}P^{-1}\log^{60} p). \tag{4.14}$$

Finally we combine the above estimates and obtain a lower bound for $I_1(N)$. For the error term in (4.9). (4.11) and (4.14) the estimate

(4.15) $$N^{3/2}P^{-1}\log^{60}p \ll U^4N^{-1/2}P^{-1/2}$$

holds because of $\delta_1 = 104\delta$. We distinguish two cases:

a) $\tilde{r} > P^{1/13}$ or $\tilde{\beta}$ does not exist. Using (4.6), (4.9), (4.10), (4.14), (4.15) and $\delta_1 = 104\delta$ we obtain for a sufficiently small δ_1

$$I_1(N) \geqq \frac{1}{2}\mathscr{P}_0\prod_p s(p) + O(U^4N^{-1/2}P^{-1/27}\log P).$$

Finally we derive from (4.4) and (4.7)

(4.16) $$I_1(N) \gg U^4N^{-1/2}.$$

b) $\tilde{r} \leqq P^{1/13}$. Using (4.6), (4.11), (4.14) and (4.15) we see

$$I_1(N) \geqq \frac{1}{2}\Omega^5\mathscr{P}_0\prod_p s(p) + O(U^4N^{-1/2}P^{-1/2}).$$

From (2.3) we conclude

$$\Omega = (1-\tilde{\beta})\log T \geqq c_3\log T(\tilde{r}^{1/2}\log^2\tilde{r})^{-1} \gg P^{-1/26}\log^{-2}P,$$

from which we deduce

(4.17) $$I_1(N) \gg U^4N^{-1/2}P^{-5/26}\log^{-10}P.$$

5. The minor arcs. Applying (1.1) we obtain

$$\sup_{\alpha\in m}|S(\alpha)| \ll U^{1+\varepsilon}\left(\frac{1}{P}+\frac{N^{1/4}}{U}+\frac{N^{2/3}}{U^2}+\frac{QN^{1/2}}{U^3}\right)^{1/4} \ll U^{1+\varepsilon}P^{-1/4}.$$

Now we can estimate $I_2(N)$ by

(5.1) $$\ll \sup_{\alpha\in m}|S(\alpha)\int_0^1|S(\alpha)|^4\,d\alpha \ll U^{1+\varepsilon}P^{-1/4}U^{2+\varepsilon} \leqq U^4N^{-1/2}P^{-3/13},$$

where in the the last step we have used

$$P^{-1/4} \leqq U^{1-2\varepsilon}N^{-1/2}P^{-3/13}.$$

This is easily seen to be correct because of $104\,\delta = \delta_1$ and thus

$$P^{\frac{-1}{4}+\frac{3}{13}} = N^{\frac{-1}{52}\delta_1} = N^{-2\delta} \leqq U^{-2\varepsilon}N^{-\delta} = U^{1-2\varepsilon}N^{-1/2}.$$

The theorem follows from (2.6), (4.16), (4.17) and (5.1).

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