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A GOLDBACH-WARING PROBLEM FOR UNEQUAL POWERS OF PRIMES

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ABSTRACT. Let p_i, $1 \leq i \leq 4$ be prime numbers. It is proved that all but $\ll x^{(1633/1680)+\varepsilon}$ even integers $n \leq x$ can be written as $n = p_1^2 + p_2^3 + p_3^4 + p_5^4$.

1. Introduction and statement of results. In 1952, Prachar [**11**] considered the following diophantine equation of the Goldbach-Waring type:

$$n = p_1^2 + p_2^3 + p_3^4 + p_4^5 \tag{1.1}$$

for prime numbers p_i, $1 \leq i \leq 4$. Defining $E(x)$ as the set of all positive even integers not larger than a real number x that cannot be written as in (1.1), Pracher showed that

$$E(x) \ll x(\log x)^{-c}, \tag{1.2}$$

for some constant $c > 0$. Roth [**15**] had earlier achieved a similar result where the p_i are allowed to take any integer value but do not necessarily have to be prime numbers. Using the approach of Montgomery and Vaughan, in [**1**] the logarithmic gain in the bound for $E(x)$ in (1.2) was replaced by a small power of x, i.e.,

$$E(x) \ll x^{1-\delta},$$

for very small $\delta > 0$. Using new techniques [**9, 10**] that allow to increase the size of the *major arcs* in the circle method and apply new estimates for the contribution of the minor arcs [**7, 8**] the upper bound for $E(x)$ was further reduced in [**2, 3, 13**] with the best bound given in [**14**] as:

$$E(x) \ll x^{(47/48)+\varepsilon}. \tag{1.3}$$

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In this paper, we further improve on these results by proving the following:

Theorem 1.
$$E(x) \ll x^{(1633/1680)+\varepsilon}.$$

We see that $1633/1680 = 1 - 47/1680$ and $47/48 = 1 - 1/48$. As $47/1680 > 1/36$, we see that Theorem 1 improves over (1.3) by a factor $> 4/3$.

We apply the circle method to prove Theorem 1. The improvement of Theorem 1 over (1.3) is mainly due to a new estimate for the integral over the *minor arcs*. Whereas previous work [**14**] used an estimate due to Roth [**15**], we develop a new estimate for the integral under consideration. We combine this estimate with a new estimate for exponential sums over powers of primes due to Kumchev [**8**]. On the *major arcs*, the iterative procedure introduced in [**9**] is combined with a new bound for Dirichlet polynomials [**4**]. This combination of techniques allows us to increase the modules of the *major arcs* compared to the procedure in [**13**], but we achieve *major arcs* slightly smaller than in [**14**].

However, the calculation of the integral over the *major arcs* is not the decisive factor to determine the size of the exceptional set $E(x)$. Indeed, the existing bounds for exponential sums over powers of primes [**8**] do not allow us to fully use the size of the *major arcs* as determined in this paper or in [**14**] in order to reduce the upper bound for $E(x)$. As we will explain later, this implies that any improvement on Theorem 1 requires further improvements of the estimates for the contribution of the integral over the *minor arcs*.

2. Notation and structure of the proof. k always denotes an integer $k \in \{2, 3, 4, 5\}$, by p we denote a prime number and L denotes $\log n$. c is a effective positive constant and ε will denote an arbitrarily small positive number; both of them may take different values at different occasions. $[a_1, \dots, a_n]$ denotes the least common multiple of the integers $a_1, \dots, a_n$. We define

$$r \sim R \Longleftrightarrow R/2 < r \leq R,$$

$$\sum_{\chi \bmod q}{}^{*} = \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}}, \qquad \sum_{1 \le a \le q}{}^{*} = \sum_{\substack{1 \le a \le q \\ (a,q)=1}}^{q}.$$

We set

(2.1)
$$P = n^{(1/12)-\varepsilon}, \quad Q = nP^{-1}L^{-E} \quad (E > 0 \text{ will be defined later}).$$

We define for any characters χ, $\chi_j \pmod q$, $q \le P$ and a fixed integer n:

$$C_k(a,\chi) = \sum_{l=1}^{q} \chi(l) e\left(\frac{al^k}{q}\right), \qquad C_k(a,\chi_0) = C_k(a,q).$$

$$Z(q,\chi_2,\chi_3,\chi_4,\chi_5) = \sum_{h=1}^{q}{}^{*} e\left(\frac{-hn}{q}\right) \prod_{k=2}^{5} C_k(h,\chi_k),$$

$$Y(q) = Z(q,\chi_0,\chi_0,\chi_0,\chi_0), \quad A(q) = \frac{Y(q)}{\phi^4(q)}.$$

When the variable n is fixed, we will always write $A(q)$ and neglect the dependency of $A(q)$ on n. Otherwise, we will write $A(q,n)$. We set $N_k :=]\sqrt[k]{n/4}, \le \sqrt[k]{n}\,]$.

$$s(p) = 1 + \sum_{\alpha \ge 1} A(p^\alpha), \quad S_k(\lambda) = \sum_{n \in N_k} \Lambda(n) e(n^k \lambda),$$

$$S_k(\lambda,\chi) = \sum_{n \in N_k} \Lambda(n)\chi(n) e(n^k\lambda), \quad T_k(\lambda) = \sum_{n \in N_k} e(n^k \lambda),$$

$$W_k(\lambda,\chi) = S_k(\lambda,\chi) - E_0 T_k(\lambda,\chi),$$

$$E_0 = \left\{ \begin{array}{ll} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{array} \right\}$$

Using the circle method we define the major arcs M and minor arcs m as follows:

$$M = \sum_{q \le P} \sum_{a=1}^{q}{}^{*} I(a,q), \qquad I(a,q) = \left[\frac{a}{q} - \frac{1}{Qq}, \frac{a}{q} + \frac{1}{Qq}\right],$$

$$m = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \Big\backslash M.$$

Let

$$R(n) = \sum_{\substack{n\in N_k,\, k\in\{2,..,5\} \\ n_2^2+..+n_5^5=n}} \Lambda(n_2)\dots\Lambda(n_5).$$

Then we find

(2.2)

$$\begin{aligned} R(n) &= \int_{1/Q}^{1+(1/Q)} e(-n\alpha)\prod_{k=2}^{5} S_k(\alpha)\,d\alpha \\ &= \left(\int_M + \int_m\right) e(-n\alpha)\prod_{k=2}^{5} S_k(\alpha)\,d\alpha =: R_1(n)+R_2(n). \end{aligned}$$

3. The integral over the minor arcs. For the estimation of the integral over the minor arcs, we use the following estimate:

Lemma 3.1.

$$\int_m |S_2(\alpha)S_3(\alpha)S_4(\alpha)S_5(\alpha)|^2 \ll n^{(47/30)-(47/1680)+\varepsilon}. \tag{3.1}$$

Proof. We denote the lefthand side of (3.1) by G. By Cauchy's inequality,

(3.2)

$$\begin{aligned} G \le &\left(\int_0^1 |S_2(\alpha)S_4^2(\alpha)|^2\,d\alpha\right)^{1/2}\left(\int_0^1 |S_2(\alpha)S_3(\alpha)S_5(\alpha)|^2\,d\alpha\right)^{1/2} \\ &\times \max_{\alpha\in m}|S_3(\alpha)|\max_{\alpha\in m}|S_5(\alpha)| := A_1^{1/2}A_2^{1/2}\max_{\alpha\in m}|S_3(\alpha)|\max_{\alpha\in m}|S_5(\alpha)|. \end{aligned}$$

The value of A_1 equals the number of integer solutions of the equation:

$$a_1^2 - a_2^2 + b_1^4 - b_2^4 + b_3^4 - b_4^4,$$

where $a_i \in N_2$, $b_i \in N_4$. The solutions are of three kinds:

1. $a_1 \ne a_2$,
2. $a_1 = a_2$, $b_1 \ne b_2$,
3. $a_1 = a_2$, $b_1 = b_2$, $b_3 = b_4$.

Thus,

$$A_1 \ll N_4^{4+\varepsilon} + N_2 N_4^{2+\varepsilon} + N_2 N_4^2 \ll n^{1+\varepsilon}. \tag{3.3}$$

The value of A_2 equals the number of integer solutions of the equation:

$$a_1^2 - a_2^2 + b_1^3 - b_2^3 + c_1^5 - c_2^5,$$

where $a_i \in N_2$, $b_i \in N_3$, $c_i \in N_5$. The solutions are of three kinds:

1. $a_1 \neq a_2$,
2. $a_1 = a_2$, $b_1 \neq b_2$,
3. $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

Thus,

$$A_2 \ll N_3^2 N_5^{2+\varepsilon} + N_2 N_5^{2+\varepsilon} + N_2 N_3 N_5 \ll n^{16/15+\varepsilon}. \tag{3.4}$$

For the reminder of the proof of Lemma 3.1, we quote two estimates from [**8**, Theorem 3]:

Lemma 3.2. a) *Suppose $\alpha \in \mathbf{R}$ and there exist $a \in \mathbf{N}$ and $q \in \mathbf{N}$ satisfying*

$$1 \le q \le n^{4/7}, \qquad (a,q) = 1, \qquad |q\alpha - a| < n^{-4/7}. \tag{3.5}$$

Then, for any fixed ε, one has

$$S_3(\alpha) \ll n^{(1/3)-(1/42)+\varepsilon} + \frac{q^\varepsilon n^{1/3} L^c}{(q + n|q\alpha - a|)^{1/2}}.$$

b) *Suppose $\alpha \in \mathbf{R}$ and there exist $a \in \mathbf{N}$ and $q \in \mathbf{N}$ satisfying*

$$1 \le q \le n^{119/216}, \qquad (a,q) = 1, \qquad |q\alpha - a| < n^{-119/216}. \tag{3.6}$$

Then, for any fixed ε, one has

$$S_5(\alpha) \ll n^{(1/5)-(1/240)+\varepsilon} + \frac{q^\varepsilon n^{1/5} L^c}{(q + n|q\alpha - a|)^{1/2}}.$$

We now consider an $\alpha \in m$ for which by Dirichlet's lemma on rational approximation we find integers a and q satisfying (3.5). By the definition of the minor arcs m, we see that either $q > P$ or $q \leq P$ and $n|q\alpha - a| \geq P$. Thus, by Lemma 3.2,

$$S_3(\alpha) \ll n^{(1/3)-(1/42)}. \tag{3.7}$$

Applying (3.6) in a similar way,

$$S_5(\alpha) \ll n^{(1/5)-(1/240)}. \tag{3.8}$$

Summarizing (3.2), (3.3), (3.4), (3.7) and (3.8), we get

$$G \ll n^{(47/30)-(47/1680)}. \qquad \square$$

We derive from Lemma 3.1 in the standard way that

$$R_2(n) \ll n^{17/60} L^{-A} \tag{3.9}$$

for any $A > 0$ and all but $(\ll x^{(1633/1680)+\varepsilon}$ even integers $3x/4 \leq n < x$. In Section 6, we will show that for any given $A > 0$,

$$R_1(n) = \frac{1}{120} P_0 \sum_{q \geq 1} A(n, q) + O\left(n^{17/60} L^{-A}\right), \tag{3.10}$$

where

$$n^{17/60} \ll P_0 := \sum_{\substack{m_2+m_3+m_4+m_5=n \\ m_k \in N_k}} \prod_{k=2}^{5} \frac{1}{m^{1-(1/k)}} \ll n^{17/60}. \tag{3.11}$$

Now Theorem 1 follows from (2.2), (3.9), (3.10), (3.11) and Lemma 4.3 below.

4. Preliminary lemmas. We will make use of the following lemmas:

Lemma 4.1. *Let $f(x)$, $g(x)$ and $f'(x)$ be three real differentiable and monotonic functions in the interval $[a,b]$. If $|f'(x)| \le \theta < 1$, $g(x), g'(x) \ll 1$, then*

$$\sum_{a<n\le b} g(n)e(f(n)) = \int_a^b g(x)e(f(x))\,dx + O\left(\frac{1}{1-\theta}\right).$$

Proof. See Lemma 4.8 in [**16**].

Lemma 4.2. *For primitive characters $\chi_i \bmod r_i$, $i = 1,2,3,4$, and the principal characters $\chi_0 \bmod q$, we have*

$$\sum_{\substack{q\le P\\ r|q}} \frac{|Z(q,\chi_0\chi_1,\chi_0\chi_2,\chi_0\chi_3,\chi_0\chi_4)|}{\phi^4(q)} \ll r^{-1+\varepsilon}(\log P)^c,$$

where $r = [r_1,r_2,r_3,r_4]$.

Proof. This is Lemma 3.3 in [**2**].

Lemma 4.3. *For any even integer n, the following is true:*

a)

$$\sum_{q>x} |A(n,q)| \ll x^{-(1/2)+\varepsilon}\, d(n).$$

b)

$$\mathcal{S}(n) := \sum_{q\ge 1} A(n,q) \gg (\log\,\log\, n)^{-c}.$$

Proof. This is Lemma 5.3 in [**13**].

5. Main lemmas. We define the following quantities for $2 \le k \le 5$:

$$I_k(g) = \sum_{r \le P} [g,r]^{-1+\varepsilon} \sum_{\chi \pmod r}{}^{*} \max_{|\lambda| \le 1/rQ} |W_k(\lambda,\chi)|,$$

$$J_k(g) = \sum_{r \le P} [g,r]^{-1+\varepsilon} \sum_{\chi \pmod r}{}^{*} \left(\int_{-1/rQ}^{1/rQ} |W_k(\lambda,\chi)|^2 d\,\lambda \right)^{1/2}.$$

The proof of Theorem 1 will make use of the following lemmas:

Lemma 5.1. *If $P \le n^{9/100-\varepsilon}$ and $g > 1$, then*

$$I_k(g) \ll g^{-1+\varepsilon} n^{(1/k)-(1/2)} L^c.$$

Lemma 5.2. *If $P \le n^{9/100-\varepsilon}$ and $g > 1$, then*

$$J_k(g) \ll g^{-1+\varepsilon} n^{1/k} L^c.$$

Lemma 5.3. *If $P \le n^{1/12-\varepsilon}$, then*

$$J_k(1) \ll n^{1/k} L^{-A},$$

for any $A > 0$.

5.1 Proof of Lemma 5.1. In order to prove the lemma we show that

$$(5.1) \quad \sum_{r \sim R} [g,r]^{-1+\varepsilon} \sum_{\chi \pmod r}{}^{*} \left(\int_{-1/rQ}^{1/rQ} |W_k(\lambda,\chi)|^2 \, d\lambda \right)^{1/2} \ll g^{-1+\varepsilon} n^{(1/k)-(1/2)} L^c$$

for $R \le P$. Applying Lemma 1, [**5**], we see

$$(5.2) \quad \int_{-1/rQ}^{1/rQ} |W_k(\lambda,\chi)|^2 \, d\lambda \ll (QR)^{-2} \cdot \int_{(n/4)-QR}^{n} \Bigg| \sum_{X < m^k \le X+Y} \Lambda(m)\chi(m) - E_0(\chi) \sum_{X < m^k \le X+Y} 1 \Bigg|^2 dt,$$

where $X = \max(t, n/4)$, $X + Y = \min(t + Qr, n)$ where $E_0(\phi) = 1$ if $\chi = \chi_0$ and $= 0$ otherwise. We first consider the case $R = 1$. A trivial estimate shows that the righthand side of (5.2) can be estimated by $\ll (QR)^{-2}n(QR)^2 n^{(2/k)-2} = n^{(2/k)-1}$, which is sufficient in view of (5.1). For $R > 1$, we argue as in [**4**]. Applying Perron's summation formula that the inner sum of (5.2) can be written as

$$(5.3) \quad S := \frac{1}{2\pi i}\int_{b-iT}^{b+iT} F_k(s,\chi)\frac{(X+Y)^{s/k} - X^{s/k}}{s}\,ds + O(T^{-1}nL^2),$$

for $T = n^2$ and $0 < b < L^{-1}$, with

$$(5.4) \qquad F_k(s,\chi) = \sum_{X^{1/k}\le m\le (X+Y)^{1/k}} \Lambda(m)\chi(m)m^{-s}.$$

Using trivial estimates, we see that for $s = \rho + it$, $0 < b < L^{-1}$,

$$\frac{(X+Y)^{s/k} - X^{s/k}}{s} \ll \min\left(T_0^{-1}, (|t|+1)^{-1}\right)$$

for $T_0 = n(QR)^{-1}$. Thus, for $b \downarrow 0$, S is bounded by

$$(5.5) \qquad \begin{aligned} S &\ll \int_{-T}^{T} |F_k(it,\chi)|\,\frac{dt}{T_0 + |t|} + \frac{L^2}{n} \\ &\ll L \max_{T_0\le T_1\le T}\frac{1}{T_1}\int_{-T_1}^{T_1} |F_k(it,\chi)|\,dt + \frac{L^2}{n}. \end{aligned}$$

Thus, we see from (5.2) and (5.5) that the lefthand side of (5.1) is upper-bounded by

$$(5.6) \quad \begin{aligned} \ll L \max_{T_0\le T_1\le T}&(QR)^{-1}T_1^{-1}n^{1/2}\sum_{r\sim R}[g,r]^{-1+\varepsilon} \\ &\sum_{\chi \pmod r}{}^{*}\int_{-T_1}^{T_1}|F_k(it,\chi)|\,dt \\ &\qquad + L^3(QR)^{-1}n^{-1/2}g^{-1}R^2. \end{aligned}$$

In view of (5.1), we see that the second term in (5.6) is permissible for $P \le n^{9/100-\varepsilon}$. We note that $[g,r](g,r) = gr$. Thus, the first term of

(5.6) is

$$\ll \max_{T_0\le T_1\le T} g^{-1+\varepsilon} L(QR)^{-1} T_1^{-1} n^{1/2} \sum_{\substack{d\le R\\ d|g}} \left(\frac{R}{d}\right)^{-1+\varepsilon} \sum_{\substack{r\sim R\\ d|r}} \sum_{\chi \pmod r}{}^{*} \int_{-T_1}^{T_1} |F_k(it,\chi)|\, dt. \tag{5.7}$$

In order to estimate the expression (5.7), we apply Theorem 1.1 from [**4**]:

Lemma 5.4. *For $F_k(s,\chi)$ defined as in* (5.4)*, there is*

$$\sum_{\substack{r\sim R\\ m|r}} \sum_{\chi} \int_{-T_1}^{T_1} |F_k(it,\chi)| \ll \left(n^{1/k} + \frac{R^2 T_1}{m} n^{11/20k}\right).$$

Applying the lemma and the estimate of divisor function $\sum_{d|g} 1 \ll g^{\varepsilon}$ to (5.7) yields

$$\begin{aligned} &\ll \max_{T_0\le T_1\le T} \max_{\substack{d|g\\ d\le R}} g^{-1+\varepsilon} L(QR)^{-1} T_1^{-1} n^{1/2} \\ &\times \left(\frac{R}{d}\right)^{-1+\varepsilon} \left(n^{1/k} + \frac{R^2 T_1}{d} n^{11/20k}\right) \\ &\ll g^{-1+\varepsilon} n^{(1/k)-(1/2)} L^c, \end{aligned}$$

for $2\le k\le 5$ and $P\le n^{9/100-\varepsilon}$, where we recall that $Q = nP^{-1}L^{-E}$.

5.2 Proof of Lemma 5.2. To prove the lemma it is enough to show that

$$\max_{R\le P/2} \max_{0\le\Delta\le(1/RQ)} \max_{\Delta\le|\lambda|\le 2\Delta} \sum_{r\sim R} [g,r]^{-1+\varepsilon} \sum_{\chi}{}^{*} |W_k(\lambda,\chi)| \ll g^{-1+\varepsilon} n^{1/k} L^C. \tag{5.8}$$

We recall the definitions (5.3) and (5.4) from the previous section and in the definition (5.4) change X to $n/4$ and $X+Y$ to n. Then we follow the argument in [**4**] and see by partial summation that

$$W_k(\lambda,\chi) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} F_k(s,\chi)V_k(s,\lambda)\,ds + O(1), \tag{5.9}$$

where $0 < b \le L^{-1}$ and $T = n^{10}$,

$$V_k(s,\lambda) = \int_{n/4}^{n} w^{s-1}e(\lambda w^k)\,dw.$$

Using well-known estimates for exponential integrals as given in [**16**, Chapter 4], we see for $\Delta \le |\lambda| \le 2\Delta$:

$$V_k(\sigma+it,\lambda) \ll n^{\sigma/k}\min\left\{T_0^{-1/2},\ \max_{n/4\le w\le n}|t+2k\pi\lambda w|^{-1}\right\}, \tag{5.10}$$

where $T_0 = n/QR$. We combine (5.9) and (5.10) and let $b \downarrow 0$, such that

$$W_k(\lambda,\chi) \ll T_0^{1/2}\int_{-T}^{T}|F_k(it,\chi)|\frac{dt}{T_0+|t|} + O(1).$$

Thus, the lefthand side of (5.8) is upper bounded by

$$\ll \max_{R\le P/2}\max_{T_0\le T_1\le T} LT_0^{1/2}T_1^{-1}\sum_{\substack{r\sim R\\ g|r}}[g,r]^{-1+\varepsilon}\sum_{\chi}\int_{-T_1}^{T_1}|F_k(it,\chi)|\,dt + \frac{R^2}{g^{1-\varepsilon}}. \tag{5.11}$$

The term $R^2/g^{1-\varepsilon}$ is permissible for $R \le n^{9/100-\varepsilon}$ in view of (5.8). For the first term, we apply Lemma 5.4 similarly to the previous section:

$$\begin{aligned}
&\ll \max_{R\le P/2}\max_{T_0\le T_1\le T}\max_{\substack{d|g\\ d\le R}} LT_0^{1/2}T_1^{-1}g^{-1+\varepsilon}\left(\frac{R}{d}\right)^{-1+\varepsilon}\\
&\quad\times\sum_{\substack{r\sim R\\ d|r}}\sum_{\chi}\int_{-T_1}^{T_1}|F_k(it,\chi)|\,dt\\
&\ll g^{-1+\varepsilon}n^{1/k}L^c,
\end{aligned}$$

for $P \le n^{9/100+\varepsilon}$.

5.3 Proof of Lemma 5.3. We treat separately the cases $R \leq L^V$ and $R \geq L^V$ for a sufficiently large V to be determined later. In the second case, we argue as in the previous section using Lemma 5.4 for $g = 1$ and use $R \geq L^V$. In the case $R \leq L^V$, we argue as in [**9**] and use the zero expansion of the von Mangoldt-function [**12**]:

$$\sum_{m \leq u} \Lambda(m)\chi(m) = E_0(\chi)u - \sum_{|\mathrm{Im}\,\rho| \leq T} \frac{u^\rho}{\rho} + O\left(\left(\frac{u}{T} + 1\right) \log^2(uT)\right),$$

where ρ runs over the nontrivial zeros of the L-function corresponding to χ mod r with $|\mathrm{Im}\,\rho| \leq T$. Thus, for $T = n^{(5/12k)-\varepsilon_1}$ and $\beta = \mathrm{Re}\,\rho$,

$$\begin{aligned} W_k(\lambda, \chi) &= \int_{\sqrt[k]{n/4}}^{\sqrt[k]{n}} e(u^k\lambda)\, d\Big\{ \sum_{n \leq u} \Lambda(m)\chi(m) - E_0(\chi)u \Big\} \\ &= \int_{\sqrt[k]{n/4}}^{\sqrt[k]{n}} e(u^k\lambda) \sum_{|\mathrm{Im}\,\rho| \leq n^{(5/12k)-\varepsilon_1}} u^{\rho-1}\, du \\ &\quad + O(n^{(7/12k)+\varepsilon_1}(1 + |\lambda|n)L^2) \\ &\ll n^{1/k} \sum_{|\mathrm{Im}\,\rho| \leq n^{(5/12k)-\varepsilon_1}} n^{(\beta-1)/k} + O(n^{1/k}L^{-A}), \end{aligned}$$

because of $P = n^{(1/12)-\varepsilon}$. We now use the fact that $L(\sigma + it, \chi)$ with χ mod r and $r \leq L^V$ has no zeros in the region, see [**12**, VIII Satz 6.2],

$$\sigma \geq 1 - \delta(T) := 1 - \frac{c_0}{\log r + (\log(T+2))^{4/5}}, \quad |t| \leq T, \tag{5.12}$$

where c_0 is an absolute constant. We also appeal to a well-known zero density estimate [**6**]:

Lemma 5.5. *Let $N^*(\alpha, T, q)$ denote the number of zeros $\sigma + it$ of all L-functions to primitive characters modulo q within the region $\sigma \geq \alpha$, $|t| \leq T$. Then for a positive integer m:*

$$\sum_{\substack{q \leq P \\ m|q}} N^*(\alpha, T, q) \ll (\log PT)^c \left(\frac{P^2T}{m}\right)^{((12/5)+\varepsilon)(1-\alpha)}.$$

Using (5.12) and Lemma 5.5, we obtain for $\delta(T) = \delta(n^{(5/12k)-\varepsilon_1}) = cL^{-4/5}$:

$$\begin{aligned}
&\max_{R\le L^V}\sum_{r\sim R} r^{-1+\varepsilon}\sum_{\chi\pmod r}{}^{*}\max_{|\lambda|\le 1/rQ}|W_k\lambda,\chi)| \\
&\ll n^{1/k}L^V\sum_{r\le L^V}\sum_{\chi \bmod r}{}^{*}\sum_{|\operatorname{Im}\rho|\le n^{(5/12k)-\varepsilon_1}} n^{(\beta-1)/k}+n^{1/k}L^{-A+2V} \\
&\ll n^{1/k}L^{V+c}\max_{0\le\beta\le 1-\delta(n^{(5/12k)-\varepsilon_1})}\big(n^{(5/12k)-\varepsilon_1}\big)^{((12/5)+\varepsilon)(1-\beta)}n^{(\beta-1)/k} \\
&\quad + n^{1/k}L^{-A+2V} \\
&\ll n^{1/k}n^{-\delta(n^{(5/12k)-\varepsilon_1})\varepsilon_1}+n^{1/k}L^{-A+2V} \\
&\ll n^{1/k}L^{-A},
\end{aligned}$$

for any $\varepsilon_1 = 2\varepsilon$ and $A =: A + 2V > 0$.

6. Treatment of the major arcs. Splitting the summation over n in residue classes modulo q, we obtain

$$S_k\bigg(\frac{a}{q}+\lambda\bigg) = \frac{C_k(a,q)}{\phi(q)}T_k(\lambda)+\frac{1}{\phi(q)}\sum_{\chi \bmod q}C_k(a,\chi)W_k(\lambda,\chi)+O(L^2).$$

Thus, we obtain from (2.2)

$$R_1(n) = R_1^m(n)+R_1^e(n)+O(n^{17/60}L^{-A}) \quad \text{(for any } A>0), \tag{6.1}$$

where

$$R_1^m(n) = \sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{1\le a\le q}{}^{*}\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}C_k(a,q)T_k(\lambda)e\Big(-\frac{a}{q}n-\lambda n\Big)d\lambda,$$

(6.2)

$$\begin{aligned}
R_1^e(n) = \sum_{k=2}^{5}\sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{1\le a\le q}{}^{*}\int_{-1/Qq}^{1/Qq}\prod_{\substack{l=2\\ l\ne k}}^{5}C_l(a,q)T_l(\lambda) \\
\times\sum_{\chi \bmod q}C_k(a,q)W_k(\lambda,\chi)\,e\Big(-\frac{a}{q}n-\lambda n\Big)d\lambda
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{k,l=2\\k<l}}^{5}\sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{1\le a\le q}{}^{*}\int_{-1/Qq}^{1/Qq}\prod_{m\in\{k,l\}}C_m(a,q)T_m(\lambda)\\
&\qquad\times\prod_{\substack{o=2\\o\ne k,l}}^{5}\sum_{\chi \bmod q}C_o(a,\chi)W_o(\lambda,\chi)\,e\Big(-\frac{a}{q}n-\lambda n\Big)d\lambda\\
&+\sum_{k=2}^{5}\sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{1\le a\le q}{}^{*}\int_{-1/Qq}^{1/Qq}C_k(a,q)T_k(\lambda)\\
&\qquad\times\prod_{\substack{l=2\\l\ne k}}^{5}\sum_{\chi \bmod q}C_l(a,q)W_l(\lambda,\chi)\,e\Big(-\frac{a}{q}n-\lambda n\Big)\,d\lambda\\
&+\sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{1\le a\le q}{}^{*}\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}\sum_{\chi \bmod q}C_k(a,\chi)W_k(\chi,\lambda)\\
&\qquad\qquad e\Big(-\frac{a}{q}n-\lambda n\Big)\,d\lambda,\\
&=: S_1+S_2+S_3+S_4.
\end{aligned}
$$

We first calculate $R_1^m(n)$. Applying Lemma 4.1 yields

$$
\begin{aligned}
T_k(\lambda)&=\int_{\sqrt[k]{n/4}}^{\sqrt[k]{n}}e(\lambda u^k)\,du+O(1)=\frac{1}{k}\int_{n/4}^{n}v^{(1/k)-1}e(\lambda v)\,dv+O(1)\\
&=\frac{1}{k}\sum_{n/4<m\le n}\frac{e(\lambda m)}{m^{1-(1/k)}}+O(1).
\end{aligned}
$$

Substituting this in $R_1^m(n)$ we see

$$
\begin{aligned}
R_1^m(n)&=\frac{1}{120}\sum_{q\le P}A(q)\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}\Big(\sum_{n/4<m\le n}\frac{e(\lambda m)}{m^{1-(1/k)}}\Big)e(-n\lambda)\,d\lambda\\
&\quad+O\Big(\sum_{p=2}^{5}\sum_{q\le P}|A(q)|\prod_{\substack{k=2\\k\ne p}}^{5}\int_{1/Qq}^{-1/Qq}\\
&\qquad\qquad\times\max\Big(\Big|\sum_{n/4<m\le n}\frac{e(\lambda m)}{m^{1-(1/k)}}\Big|,1\Big)\,d\lambda\Big).
\end{aligned}
$$

Using Lemmas 4.2 and 4.3 as well as the trivial bound

$$\sum_{n/4<m\le n}\frac{e(\lambda m)}{m^{1-(1/k)}}\ll \min\left(\sqrt[k]{n},\frac{1}{n^{1-(1/k)}|\lambda|}\right), \tag{6.3}$$

we obtain
(6.4)

$$\begin{aligned}R_1^m(n)=&\frac{1}{120}\sum_{q\le P}A(q)\int_{-1/2}^{1/2}\prod_{k=2}^{5}\left(\sum_{n/4<m\le n}\frac{e(\lambda m)}{m^{1-(1/k)}}\right)e(-n\lambda)\,d\lambda\\&+O\left(\sum_{q\le P}|A(q)|\int_{1/Qq}^{1/2}\frac{1}{n^{3-(17/60)}|\lambda|^4}\,d\lambda\right)+O(n^{(17/60)}L^{-A})\\=&\frac{1}{120}P_0\sum_{q\le P}A(q)+O((PQ)^3n^{(17/60)-3}L^c)\\&+O(n^{17/60}L^{-A})\\=&\frac{1}{120}P_0\sum_{q\ge 1}A(q)+O(n^{17/60}L^{-A}),\end{aligned}$$

where P_0 is defined as in (3.11) and E is chosen sufficiently large in $Q=nP^{-1}L^{-E}$. Applying Lemma 4.2, we can estimate S_4 in the following way:

(6.5)

$$\begin{aligned}|S_4|=&\left|\sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{\chi_1\bmod q}\sum_{\chi_2\bmod q}\sum_{\chi_3\bmod q}\sum_{\chi_4\bmod q}Z(q,\chi_1,\chi_2,\chi_3,\chi_4)\right.\\&\left.\times\int_{-1/Qq}^{1/Qq}\prod_{j=1}^{4}W_{j+1}(\lambda,\chi_j)e(-n\lambda)d\lambda\right|\\\le&\sum_{r_1\le P}\sum_{r_2\le P}\sum_{r_3\le P}\sum_{\substack{r_4\le P\\ [r_1,r_2,r_3,r_4]\le P}}\\&\times\sum_{\chi_1\bmod r_1}{}^{*}\sum_{\chi_2\bmod r_2}{}^{*}\sum_{\chi_3\bmod r_3}{}^{*}\sum_{\chi_4\bmod r_4}{}^{*}\\&\times\int_{-1/Q[r_1,r_2,r_3,r_4]}^{1/Q[r_1,r_2,r_3,r_4]}\prod_{j=1}^{4}|W_{j+1}(\lambda,\chi_j)|\,d\lambda\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{\substack{q\le P\\ [r_1,r_2,r_3,r_4]|q}} \frac{|Z(q,\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0\chi_4\chi_0)|}{\phi^4(q)}\\
&\ll L^c \sum_{r_1\le P}\sum_{r_2\le P}\sum_{r_3\le P}\sum_{r_4\le P}[r_1,r_2,r_3,r_4]^{-1+\varepsilon}\\
&\times \sum_{\chi_1 \bmod r_1}{}^{*}\sum_{\chi_2 \bmod r_2}{}^{*}\sum_{\chi_3 \bmod r_3}{}^{*}\\
&\qquad\qquad \times \sum_{\chi_4 \bmod r_4}{}^{*}\int_{-1/Q[r_1,r_2,r_3,r_4]}^{1/Q[r_1,r_2,r_3,r_4]}\prod_{j=1}^{4}|W_{j+1}(\lambda,\chi_j)|\,d\lambda\\
&\le L^c \sum_{r_1\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\max_{|\lambda|\le 1/r_1Q}|W_2(\lambda,\chi_1)|\\
&\times \sum_{r_2\le P}\sum_{\chi_2 \bmod r_2}{}^{*}\max_{|\lambda|\le 1/r_2Q}\sum_{r\le P}\sum_{\chi}{}^{*}|W_3(\lambda,\chi_2)|\\
&\times \sum_{r_3\le P}\sum_{\chi_3 \bmod r_3}{}^{*}\left(\int_{-1/rQ}^{1/r_3Q}|W_4(\lambda,\chi_3)|^2\,d\lambda\right)^{1/2}\\
&\times \sum_{r_4\le P}\sum_{\chi_4 \bmod r_4}{}^{*}[r_1,r_2,r_3,r_4]^{-1+\varepsilon}\\
&\times \left(\int_{-1/Qr_4}^{1/Qr_4}|W_5(\lambda,\chi_4)|^2\,d\lambda\right)^{1/2}.
\end{aligned}
$$

Since $[r_1,r_2,r_3,r_4]=[[r_1,r_2,r_3],r_4]$, we use Lemma 5.1 to estimate the sum over r_4:

$$
\begin{aligned}
\sum_{r_4\le P}[r_1,r_2,r_3,r_4]^{-1+\varepsilon}&\sum_{\chi_4 \bmod r_4}{}^{*}\left(\int_{-1/r_4Q}^{1/r_4Q}|W_5(\lambda,\chi_4)|^2\,d\lambda\right)^{1/2}\\
&= I_5([r_1,r_2,r_3]) \ll [r_1,r_2,r_3]^{-1+\varepsilon}n^{-3/10+\varepsilon}L^c.
\end{aligned}
$$

Applying Lemma 5.1 again we see that the contribution of this quantity to the sum over r_3 is:

$$\ll n^{-3/10+\varepsilon} L^c \sum_{r_3 \le P} [r_1, r_2, r_3]^{-1+\varepsilon} \sum_{\chi_3 \bmod r_3}{}^{*} \left(\int_{-1/r_3 Q}^{1/r_3 Q} |W_4(\lambda, \chi_3)| \, d\lambda \right)^{1/2}$$
$$\le n^{-3/10+\varepsilon} I_4([r_1, r_2]) L^c \ll [r_1, r_2]^{-1+\varepsilon} n^{-11/20+\varepsilon} L^c.$$

Applying Lemma 5.2, we see

$$\ll n^{-11/20+\varepsilon} L^c \sum_{r_2 \le P} [r_1, r_2]^{-1+\varepsilon} \sum_{\chi_2 \bmod r_2}{}^{*} \max_{|\lambda| \le 1/r_2 Q} |W_3(\lambda, \chi_2)|$$
$$\le n^{-11/20+\varepsilon} L^c J_3([r_1]) \ll [r_1]^{-1+\varepsilon} n^{-13/60} L^c.$$

Finally, we apply Lemma 5.3 to estimate the sum over r_1 as follows:

$$S_4 \ll n^{-13/60} L^c \sum_{r_1 \le P} r_1^{-1+\varepsilon} \sum_{\chi_1 \bmod r_1}{}^{*} \max_{|\lambda| \le 1/r_1 Q} |W_2(\lambda, \chi_1)|$$
$$= n^{-13/60} L^c J_2(1) \ll n^{17/60} L^{-A}.$$

For the estimation of the sums $\sum_2 - \sum_4$ we note that

$$T = \max_{|\lambda| \le 1/Q} |T_k(\lambda)| \ll n^{1/k},$$

and using (6.3) we get:

$$\left(\int_{-1/Q}^{1/Q} |T_k(\lambda)|^2 \, d\lambda \right)^{1/2} \ll n^{(1/k)-(1/2)}.$$

Using these estimates and the Lemmas 5.1–5.3, we argue similarly to the estimation of S_4 and obtain:

$$S_1 + S_2 + S_3 \ll n^{17/60} L^{-A}. \tag{6.6}$$

Thus, we see from (6.1), (6.2), (6.4), (6.5) and (6.6):

$$R_1^m(n) = \frac{1}{120} P_0 \sum_{q \geq 1} A(q) + O(n^{17/60} L^{-A}),$$

i.e., we proved (3.10). □

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