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THE TERNARY GOLDBACH CONJECTURE WITH PRIMES IN THIN SUBSETS

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ABSTRACT. First, we construct a thin subset S of primes, satisfying

$$|S \cap [1,x]| \ll x^{26/29+\varepsilon}$$

such that every sufficiently large odd integer N can be represented as

$$\begin{cases} N = p_1 + p_2 + p_3, \\ p_j \in S & j = 1,2,3. \end{cases}$$

Second, let k be a prime number and b_j positive integers with $(b_j,k) = 1$, $j = 1,2,3$. We show that, for any $F > 0$, and for all but $O(R(\log R)^{-F})$ prime moduli $k \le R = N^{(9/58)-\varepsilon}$, every sufficiently large odd integer $N = b_1 + b_2 + b_3 \pmod k$ can be written as

$$\begin{cases} N = p_1 + p_2 + p_3, \\ p_j \equiv b_j \pmod k & j = 1,2,3. \end{cases}$$

1. Introduction. In 1937, Vinogradov [**16**] proved that every sufficiently large odd number can be written as the sum of three prime numbers. Many researchers have considered the three primes problem by imposing restrictions on the choice of the three primes. In 1986, Wirsing [**17**] proved the existence of a thin subset S of the set of prime numbers that satisfies

$$|S \cap [1,x]| \ll (x \log x)^{1/3}$$

such that every sufficiently large integer N can be written as

$$\begin{cases} N = p_1 + p_2 + p_3, \\ p_j \in S & j = 1,2,3. \end{cases} \tag{1.1}$$

Wirsing's result is based on probabilistic arguments, such that the set S is not constructed explicitly. In [**2**], a first attempt on constructing

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explicit thin subsets was made using Piatetski-Shapiro primes. This approach was later also applied to the binary Goldbach conjecture [**8**]. A different approach was introduced by Wolke [**18**] who constructed a set S explicitly which satisfies

$$|S \cap [1, x]| \ll x^{15/16}.$$

Adding new ideas to Wolke's approach, Liu [**11**] improved on Wolke's result by constructing a set S that satisfies

$$|S \cap [1, x]| \ll x^{9/10} \log^c x,$$

for a fixed $c > 0$. In this paper, we improve on the result of Liu by proving the following theorem:

Theorem 1. *There exists an explicitly known subset S of the set of primes such that $|S \cap [1, x]| \ll x^{26/29+\varepsilon}$, where ε is an arbitrarily small positive number, such that every sufficiently large odd integer N can be written in the form* (1.1)*.*

The proof of this theorem relies on another generalization of the ternary Goldbach conjecture. We consider the three primes problem where we require the primes to belong to pre-defined arithmetic progressions to a fixed module. In 1953, Ayoub [**1**] proved the following result:

If k is a fixed positive integer, b_i, $i = 1, 2, 3$, are integers with $(b_i, k) = 1$ and $J(N; k, b_1, b_2, b_3)$ is the number of solutions of the equation

$$\begin{cases} N = p_1 + p_2 + p_3, \\ p_j \equiv b_j \pmod{k}, \end{cases} \tag{1.2}$$

then

$$J(N; k, b_1, b_2, b_3) = \sigma(N; k) \frac{N^2}{2 \log^3 N} (1 + o(1)),$$

where for odd integer $N \equiv b_1 + b_2 + b_3 \pmod{k}$*,*

$$\sigma(N,k) = \frac{C(k)}{k^2} \prod_{p|k} \frac{p^3}{(p-1)^3+1} \prod_{\substack{p|N \\ p\nmid k}} \frac{(p-1)((p-1)^2-1)}{(p-1)^3+1} \times \prod_{p>2} \left(1 + \frac{1}{(p-1)^3}\right),$$

where all $p > 2$*,* $C(k) = 2$ *for odd* k *and* $C(k) = 8$ *for even* k*.* Ayoub's method allows to prove this result for all $k \leq \log^A N$ for an arbitrary $A > 0$ for all sufficiently large odd integers N. Later, Liu and Zhan [**14**] as well as the author [**4**] showed the solvability of the equation (1.2) for larger moduli k :

For an odd $N \equiv b_1 + b_2 + b_3 \pmod{k}$ *and* N *sufficiently large, there holds*

$$J(N;k,b_1,b_2,b_3) > 0 \tag{1.3}$$

for all $k \leq N^\delta$*, where* δ *is a very small, positive constant.*

In [**13**], it was shown that (1.2) holds for all $k \leq R = N^{(1/8)-\varepsilon}$, $\varepsilon > 0$, with at most $O(R(\log R)^{-A})$ exceptions for any $A > 0$. Earlier, Wolke [**18**] had proved that if k is restricted to be a prime number, R can be chosen as large as $N^{1/11}(\log N)^{-A}$ for any $A > 0$ with at most $O(R(\log R)^{-A})$ exceptions for any $A > 0$, respectively. Liu [**11**] later improved on this result by choosing R as large as $N^{3/20}(\log N)^{-A}$. In [**5**], the exceptional set from [**11**] was further reduced to $O((\log N)^B)$ prime moduli k at the cost of a smaller upper bound $R = N^{(5/48)-\varepsilon}$. In this paper, we increase the value of k from Liu's value $N^{3/20}(\log N)^{-A}$ to $N^{(9/58)-\varepsilon}$ for all but $O(R(\log R)^{-A})$ prime modules $k \leq R$:

Theorem 2. *Let* $R = N^{9/58-\varepsilon}$*. Then the inequality* (1.3) *holds for all prime numbers* $k \leq R$ *with at most* $O(R(\log R)^{-A})$ *exceptions for any* $A > 0$*.*

Using that $\sigma(N,k) \gg 1/k^2$, Theorem 2 follows via a dyadic argument directly from the following theorem.

Theorem 3. *Let $R = N^{9/58-\varepsilon}$ and $A > 0$. Then there exists $B = B(A) > 0$ such that*

$$\sum_{k\sim R} k \max_{(b_i,k)=1} \left| \sum_{\substack{N=n_1+n_2+n_3 \\ n_i\equiv b_i \pmod k}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) - \sigma(N,k)\frac{N^2}{32} \right| \ll N^2(\log N)^{-A},$$

where the summation over k only includes prime numbers k, and we use the abbreviation $r \sim R \Leftrightarrow R/2 < r \leq R$.

The increase of the size of R compared to previous work is due to the application of two techniques. As in previous works, we split the major arcs into two sets: the first set contains Farey sections around midpoints a/q where $k \nmid q$, whereas in the second set $k|q$. For $k|q$, we apply an estimate from [**11**]. For $k \nmid q$, we use a technique first applied by Liu in [**12**] to the Lagrange theorem with prime variables. The technique is a further development of an approach explained in [**12**] which was first applied to the Goldbach conjecture with primes in arithmetic progressions in [**5**] and allowed to choose $R = k^{5/48-\varepsilon}$.

The proof of Theorem 1 is omitted, since its proof follows the proof of the main theorem in [**18**] by using [**18**, Theorem 2 and Lemma 2].

2. Outline of the proof of Theorem 3 and treatment of the minor arcs. In the sequel, $[a_1, \dots, a_n]$ denotes the least common multiple of the integers $a_1, \dots, a_n$, c is an effective positive constant and ε will denote an arbitrarily small positive number; both of them may take different values at different occasions. For example, we may write

$$L^c L^c \ll L^c, \quad N^\varepsilon L^c \ll N^\varepsilon.$$

We use the familiar notations

$$\sum_{\chi \bmod q}{}^{*} := \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}}, \qquad \sum_{1\leq a\leq q}{}^{*} := \sum_{\substack{1\leq a\leq q \\ (a,q)=1}}.$$

We know from [**1**] that Theorem 3 holds true for $k \leq L^H$ for any $H > 0$. Therefore, we assume throughout the document that

$$k > L^H \tag{2.1}$$

for a fixed $H > 0$ to be determined later. χ_q denotes a character modulo q and $\chi_{q,0}$ is the principal character modulo q. We write $e(\alpha) = e^{2\pi i\alpha}$ and the variables p and k always denote prime numbers. We keep k fixed throughout this paper. If $p^m|q$, but $p^{m+1} \nmid q$, we write $p^m \,||\, q$. We define for any three positive integers r_i, $i \in \{1, 2, 3\}$, that satisfy $k^3 \nmid r_i$:

$$s_i = \begin{cases} r_i & \text{if } k \not| r_i, \\ r_i/k & \text{if } k||r_i, \\ r_i/k^2 & \text{if } k^2||r_i. \end{cases} \tag{2.2}$$

Setting $[r_1, r_2, r_3] = r$ and $[s_1, s_2, s_3] = s$, this implies for $k^m||r$, $m \le 2$:

$$r = sk^m. \tag{2.3}$$

For a positive integer q and a character χ modulo q, let

$$k_q = (k, q), \quad R(N) = \sum_{\substack{N/4 \le n_i \le N \\ n \equiv b_i \pmod{k} \\ n_1+n_2+n_3=N}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3), \tag{2.4}$$

$$C(\chi, q, h, b, a) = \sum_{\substack{m=1 \\ m \equiv b \pmod{h}}}^{q} \chi(m) e\left(\frac{ma}{q}\right),$$

$$Z(N, q, k_q, \chi_1, \chi_2, \chi_3) := \frac{1}{\phi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C(\chi_1, q, k_q, b_1, a)$$

$$\times C(\chi_2, q, k_q, b_2, a) C(\chi_3, q, k_q, b_3, a) e\left(\frac{-aN}{q}\right),$$

$$A(N, q, k_q) = Z\left(N, q, k_q, \chi_{(q/k_q),0}, \chi_{(q/k_q),0}, \chi_{(q/k_q),0}\right),$$

$$T(\lambda) = \sum_{N/4 \le n \le N} e(\lambda n).$$

As we always argue for a fixed variable N, we define

$$
\begin{aligned}
S(\lambda,k,b_i) &= \sum_{\substack{N/4\le n\le N\\ n\equiv b_i \pmod k}} \Lambda(n)e(n\lambda),\\
S(\lambda,\chi) &= \sum_{N/4\le n\le N} \Lambda(n)e(n\lambda)\chi(n),\\
W(\lambda,\chi) &= S(\lambda,\chi) - E_0(\chi)T(\lambda),\\
E_0(\chi) &= \begin{cases} 1 & \text{if } \chi=\chi_0,\\ 0 & \text{otherwise,}\end{cases}\\
P_1 = k^{4/3}L^{3G}, &\qquad P_2 = k^2L^{3G},\\
Q = Nk^{-2}L^{-4G}, &\qquad Q_1 = NL^{-H},
\end{aligned} \tag{2.5}
$$

where the constants $G\ge 8$ and H will be specified later. Using the circle method, we define the major arcs $M = E_1(k)\cup E_2(k)$ as in [**9**]:

$$
E_1(k) = \bigcup_{\substack{q\le P_1\\ k\nmid q}} \bigcup_{\substack{a=1\\ (a,q)=1}}^{q} \left[\frac{a}{q}-\frac{1}{qQ}, \frac{a}{q}+\frac{1}{qQ}\right],
$$

$$
E_2(k) = \bigcup_{\substack{q\le P_2\\ k| q}} \bigcup_{\substack{a=1\\ (a,q)=1}}^{q} \left[\frac{a}{q}-\frac{1}{qQ}, \frac{a}{q}+\frac{1}{qQ}\right].
$$

We define the minor arcs m as $m = [1/Q,\, 1+(1/Q)]\setminus M$. Using Dirichlet's theorem on rational approximation, we find that $m \subset E_3(k)\cup E_4(k)$, where

$$
E_3(k) = \left\{\alpha = \frac{a}{q}+\lambda \,:\, P_1<q<Q,\ k\nmid q,\ |\lambda|\le \frac{1}{qQ}\right\},
$$

$$
E_4(k) = \left\{\alpha = \frac{a}{q}+\lambda \,:\, P_2<q<Q,\ k|q,\ |\lambda|\le \frac{1}{qQ}\right\}.
$$

In order to prove Theorem 3, it is obviously sufficient to show

$$
\sum_{k\sim R} k\left|\int_0^1 S(\alpha,k,b_1)S(\alpha,k,b_2)S(\alpha,k,b_3)e(-\alpha N)d\alpha - \sigma(N,k)\frac{N^2}{32}\right| \ll N^2L^{-A}. \tag{2.6}
$$

We now recall the following lemma from [**5**].

Lemma 2.1.

$$\left(\sum_{\substack{q\le P_1\\ k\nmid q}}+\sum_{\substack{q\le P_2\\ k|q}}\right)\frac{1}{\phi\left(k/k_q\right)^3}A(N,q,k_q)=\sigma(N,k)+O(Pk)^{-1}L^c.$$

Proof. See [**5**, Lemma 3.5]. □

Using Lemma 2.1, the estimate (2.6) can now be shown by proving the following three estimates:

$$\sum_{k\sim R}k\left|\int_{E_1(k)}S(\alpha,k,b_1)S(\alpha,k,b_2)S(\alpha,k,b_3)e(-\alpha N)\,d\alpha\right.$$
$$\left.-\frac{1}{\phi^3\left(k\right)}\sum_{\substack{q\le P_1\\ k\nmid q}}A(N,q,1)\frac{N^2}{32}\right|\ll N^2L^{-A}, \tag{2.7}$$

$$\sum_{k\sim R}k\left|\int_{E_2(k)}S(\alpha,k,b_1)S(\alpha,k,b_2)S(\alpha,k,b_3)e(-\alpha N)\,d\alpha\right.$$
$$\left.-\sum_{\substack{q\le P_2\\ k|q}}A(N,q,k)\frac{N^2}{32}\right|\ll N^2L^{-A}, \tag{2.8}$$

$$\sum_{k\sim R}k\left|\int_{E_3(k)\cup E_4(k)}S(\alpha,k,b_1)S(\alpha,k,b_2)S(\alpha,k,b_3)e(-\alpha N)\,d\alpha\right|$$
$$\ll N^2L^{-A}, \tag{2.9}$$

where the summation over k is over k prime only.

In order to estimate the contribution over the minor arcs, we apply an estimate for exponential sums over primes in arithmetic progressions [**3**] and obtain in the same way as in [**5**]:

(2.10)
$$\int_{E_3(k)\cup E_4(k)} S(\alpha,k,b_1)S(\alpha,k,b_2)S(\alpha,k,b_3)e(-\alpha N)\,d\alpha \ll N^2k^{-2}L^{-A}.$$

The estimate (2.10) immediately implies the estimate (2.9).

3. Preliminary lemmas.

Lemma 3.2. *Let $f(x)$, $g(x)$ and $f'(x)$ be three real differentiable and monotonic functions in the interval $[a,b]$ and $|g(x)| \ll M$.*

(i) *If $|f'(x)| \gg m > 0$, then*
$$\int_a^b g(x)\,e(f(x))\,dx \ll M/m.$$

(ii) *If $|f''(x)| \gg r > 0$, then*
$$\int_a^b g(x)\,e(f(x))\,dx \ll M/r^{1/2}.$$

Proof. See [**15**, Chapter 21]. □

Lemma 3.2. *Let there be primitive characters $\chi_i \bmod r_i$, $i = 1,2,3$, the principal character $\chi_0 \bmod q$ and $r = [r_1,r_2,r_3]$.*

If $(r,k) = 1$, then
$$\sum_{\substack{q\le P\\ r|q}} |Z(N,q,k_q,\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0)| \ll r^{-1/2}L.$$

Proof. See [**5**, Lemma 3.4]. □

4. Lemmas to calculate the integrals over $E_1(k)$. We define the following quantities which we will need for the proof of Theorem 3:
$$J^A = k^{-1/9} \sum_{\substack{r\le P_1k\\ k||r}} (r/k)^{-1/2+\varepsilon} \sum_{\chi \pmod r}{}^{*} \max_{|\lambda|\le k/rQ} |W(\lambda,\chi)|,$$

$$K^A(g) = k^{-4/9} \sum_{\substack{r \le P_1 k \\ k||r}} [g, r/k]^{-1/2+\varepsilon} \sideset{}{^*}\sum_{\chi (\operatorname{mod} r)} \left(\int_{-k/rQ}^{k/rQ} |W(\lambda, \chi)|^2 \, d\lambda \right)^{1/2}.$$

$$J^B = k^{-1/9} \sum_{\substack{r \le P_1 \\ k \nmid r}} r^{-1/2+\varepsilon} \sideset{}{^*}\sum_{\chi (\operatorname{mod} r)} \max_{|\lambda| \le 1/rQ} |W(\lambda, \chi)|,$$

$$K^B(g) = k^{-4/9} \sum_{\substack{r \le P_1 \\ k \nmid r}} [g, r]^{-1/2+\varepsilon} \sideset{}{^*}\sum_{\chi (\operatorname{mod} r)} \left(\int_{-1/rQ}^{1/rQ} |W(\lambda, \chi)|^2 \, d\lambda \right)^{1/2}.$$

The proof of Theorem 3 will make use of the following lemmas.

Lemma 4.1. *If* $k \le N^{9/58-\varepsilon}$, *then for* $F \in \{A, B\}$,

$$K^F(g) \ll g^{-1/2+\varepsilon} N^{1/2} L^c.$$

Lemma 4.2. *If* $k \le N^{9/58-\varepsilon}$, *then for* $F \in \{A, B\}$,

$$J^F \ll N L^{-A},$$

for any $A > 0$.

5. Proof of Lemma 4.1.

5.1 Proof of Lemma 4.1 **for** $K^A(g)$**.** In order to prove the lemma, we show that

$$\sum_{\substack{r \sim R \\ k||r}} [g, r/k]^{-1/2+\varepsilon} \sideset{}{^*}\sum_{\chi \pmod r} \left(\int_{-k/rQ}^{k/rQ} |W(\lambda, \chi)|^2 \, d\lambda \right)^{1/2} \ll g^{-1/2+\varepsilon} N^{1/2} k^{4/9} L^c \tag{5.1}$$

for $R \le P_1 k$. Applying Lemma 1, [**6**], we see

$$\int_{-k/rQ}^{k/rQ} |W(\lambda, \chi)|^2 \, d\lambda \ll (QR/k)^{-2} \int_{N/8}^{N} \left| \sum_{\substack{t < m \le t + Qr/k \\ N/4 < m \le N}} \Lambda(m) \chi(m) \right|^2 dt, \tag{5.2}$$

We note that $E_0(\chi) = 0$ because of $r \geq k$ and the primitivity of the characters. We set $X = \max(N/4, t)$ and $X + Y = \min(N, t + Qr/k)$. We apply a slight modification of Heath-Brown's identity [**7**]

$$-\frac{\zeta'}{\zeta}(s) = \sum_{j=1}^{K} \binom{K}{j} (-1)^j \zeta'(s)\zeta^{j-1}(s) M^j(s) - \frac{\zeta'}{\zeta}(s)(1 - \zeta(s)M(s))^K,$$

with $K = 5$ and

$$M(s) = \sum_{n \leq N^{1/5}} \mu(n) n^{-s}$$

to the sum

$$\sum_{X < m \leq X+Y} \Lambda(m)\chi(m).$$

Arguing exactly as in [**10**], we find by applying Heath-Brown's identity and Perron's summation formula that the inner sum of (5.2) is a linear combination of $O(L^c)$ terms of the form

$$S_{I_{a_1},\dots,I_{a_{10}}} = \frac{1}{2\pi i} \int_{-T}^{T} F\left(\tfrac{1}{2} + iu, \chi\right) \frac{(X+Y)^{((1/2)+iu)} - X^{((1/2)+iu)}}{1/2+iu} \, du$$
$$+ O(T^{-1} N L^2),$$

where $2 \leq T \leq N$,

$$F(s, \chi) = \prod_{j=1}^{10} f_j(s, \chi),$$
$$f_j(s, \chi) = \sum_{n \in I_j} a_j(n) \chi(n) n^{-s},$$
$$a_j(n) = \begin{cases} \log n \text{ or } 1 & j = 1, \\ 1 & 1 < j \leq 5, \\ \mu(n) & 6 \leq j \leq 10. \end{cases}$$
$$I_j = (N_j,\, 2N_j], \quad 1 \leq j \leq 10,$$
$$N \ll \prod_{j=1}^{10} N_j \ll N, \quad N_j \leq N^{1/5}, \quad 6 \leq j \leq 10.$$

Since

$$\frac{(X+Y)^{(1/2+iu)} - X^{(1/2+iu)}}{1/2+iu} \ll \min\left(QRk^{-1}N^{-1/2}, N^{1/2}\left(|u|+1\right)^{-1}\right)$$

by taking $T = N$ and $T_0 = N(QR/k)^{-1}$, we conclude that for a sufficiently large G, $S_{I_{a_1},\cdots,I_{a_{10}}}$ is bounded by

$$\ll QRk^{-1}N^{-1/2}\int_{-T_0}^{T_0}\left|F\left(\frac{1}{2}+iu,\chi\right)\right|du \\ + N^{1/2}\int_{T_0\le|u|\le T}\left|F\left(\frac{1}{2}+iu,\chi\right)\right|\frac{du}{|u|} + L^2.$$

Thus, we derive from (5.2) that in order to prove (5.1), it is enough to show that for $R \le P_1 k$:

$$\sum_{\substack{r\sim R\\ k||r}}[g,r/k]^{-1/2+\varepsilon}\sum_{\chi}{}^{*}\int_{0}^{T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt \\ \ll g^{-1/2+\varepsilon}N^{1/2}k^{4/9}L^C, \tag{5.3}$$

$$\sum_{\substack{r\sim R\\ k||r}}[g,r/k]^{-1/2+\varepsilon}\sum_{\chi}{}^{*}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt \\ \ll N^{-1/2}QRk^{-5/9}T_1L^C, \quad T_0 < |T_1| \le T. \tag{5.4}$$

For the proof of (5.3) and (5.4) we quote the following lemma which is shown for $m = 1$ in [**12**, Lemma 5.2] and for the general case $m \ge 1$ in [**11**, Lemma 2.1].

Lemma 5.1. *Let $F(s,\chi)$ be defined as above. Then for any $R \ge 1$ and $T_2 > 0$,*

$$\sum_{\substack{r\sim R\\ m|r}}\sum_{\chi}{}^{*}\int_{T_2}^{2T_2}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt \\ \ll \left(\frac{R^2}{m}T_2 + \frac{R}{m^{1/2}}T_2^{1/2}N^{3/10} + N^{1/2}\right)L^c.$$

Now, for the proof of (5.3), we note that $[g, r/k](g, r/k) = gr/k$. Thus, we see from Lemma 5.1 that the lefthand side of (5.3) is

$$\ll g^{-1/2+\varepsilon} \sum_{\substack{d \le R/k \\ d|g}} \left(\tfrac{R}{kd}\right)^{-1/2+\varepsilon} \sum_{\substack{r \sim R \\ d|(r/k),\, k|r}} \sum_{\chi \pmod r}{}^{*} \int_0^{T_0} \left|F\left(\tfrac{1}{2}+it,\chi\right)\right| dt$$

$$\ll g^{-1/2+\varepsilon} \sum_{\substack{d \le R/k \\ d|g}} \left(\tfrac{R}{kd}\right)^{-1/2+\varepsilon} \left(\tfrac{R^2}{kd} T_0 + \tfrac{R}{k^{1/2}d^{1/2}} T_0^{1/2} N^{3/10} + N^{1/2}\right) L^c$$

$$\ll g^{-1/2+\varepsilon} N^{1/2} k^{4/9} L^c$$

for $k \le N^{(9/58)-\varepsilon}$, where we have used (2.5). Equation (5.4) is shown in the same way.

5.2. Proof of Lemma 4.1 **for** $K^B(g)$**.** Arguing analogously to subsection 5.1, we find that the proof of Lemma 4.1 for $F = B$ reduces to the proof of the following two estimates: For $T = N$, $T_0 = N(QR)^{-1}$, $R \le P_1$, and $k \le N^{9/40-\varepsilon}$, the following must hold.

$$\sum_{r \sim R} [g, r]^{-1/2+\varepsilon} \sum_{\chi}{}^{*} \int_0^{T_0} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll g^{-1/2+\varepsilon} N^{1/2} k^{4/9} L^c,$$

$$\sum_{r \sim R} [g, r]^{-1/2+\varepsilon} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt$$
$$\ll g^{-1/2+\varepsilon} N^{-1/2} QRk^{-5/9} T_1 L^c, \quad T_0 < |T_1| \le T.$$

These estimates follow from Lemma 5.1 in the same way as in the proof of $K^A(g)$.

6. Proof of Lemma 4.2.

6.1. Proof of Lemma 4.2 **for** $J^A(g)$**.** To prove the lemma, it is enough to show that

(6.1)
$$\max_{R \le P_1 k} \sum_{\substack{r \sim R \\ k||r}} \left(\frac{r}{k}\right)^{-1/2+\varepsilon} \sum_{\chi \pmod r}{}^{*} \max_{|\lambda| \le k/rQ} |W(\lambda,\chi)| \ll N k^{1/9} L^{-A}.$$

Arguing as in the section before (we do not have to apply Gallagher's lemma here) we find

$$W(\lambda,\chi) \ll L^c \max_{I_{a_1},\cdots,I_{a_{10}}} \Bigg| \int_{-T}^{T} F\left(\frac{1}{2}+it,\chi\right) \times \int_{N/4}^{N} u^{-1/2} e\left(\frac{t}{2\pi}\log u + \lambda u\right) du\, dt \Bigg| + L^2 k^3.$$

Here, we have set $T = N$ and used that $|\lambda| \leq k/rQ$. Estimating the inner integral by Lemma 3.2, we obtain

$$\int_{N/4}^{N} u^{-(1/2)} e\left(\frac{t}{2\pi}\log u + \lambda u\right) du \ll N^{-1/2} \min\left(\frac{N}{\sqrt{|t|+1}}, \frac{N}{\min_{N/2<u\leq N}|t+2\pi\lambda u|}\right).$$

Taking $T_0 = 4\pi N(QR/k)^{-1}$, we conclude that in order to prove (6.1) it is enough to prove that

$$\sum_{\substack{r\sim R\\ k||r}} \left(\frac{r}{k}\right)^{-1/2+\varepsilon} \sum_{\chi}{}^{*} \int_{0}^{T_0} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}(T_0+1)^{1/2} k^{1/9} L^c,$$

$$\sum_{\substack{r\sim R\\ k||r}} \left(\frac{r}{k}\right)^{-1/2+\varepsilon} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2} k^{1/9} T_1 L^c, \quad T_0 < |T_1| \leq T.$$

Using (2.1) with a sufficiently large $H = H(A)$ and (2.5), these estimates follow from Lemma 5.1 for $k \leq N^{9/58-\varepsilon}$.

6.2. Proof of Lemma 4.2 **for** $\mathbf{J}^B(g)$**.** Throughout this section we set $T = N$ and $T_0 = N(QR)^{-1}$. Arguing as in subsection 6.1, we see that to estimate J^B it is enough to show that for $k \le N^{9/40-\varepsilon}$ and $R \le P_1$, we have

$$\sum_{r\sim R} r^{-1/2+\varepsilon} \sum_{\chi}{}^{*} \int_0^{T_0} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}(T_0+1)^{1/2}k^{1/9}L^{-A},$$

$$\sum_{r\sim R} r^{-1/2+\varepsilon} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}k^{1/9}T_1L^{-A}, \quad T_0 < |T_1| \le T.$$

Both estimates follow straight from (2.1) and Lemma 5.1.

7. The major arcs.

7.1. The contribution of the interval over $E_1(k)$. For $k \nmid q$, we find

$$S\left(\frac{a}{q}+\lambda, b_i\right) = \sum_{g=1}^{q}{}^{*} e\left(\frac{ga}{q}\right) \sum_{\substack{N/4<n\le N \\ n\equiv b_i \pmod k \\ n\equiv g \pmod q}} \Lambda(n)\, e(n\lambda) + O(L^2).$$

We shall introduce the Dirichlet characters ξ mod k and χ mod q and obtain, as in [**5, 13**] for $k \not| q$:

$$\begin{aligned} S\left(\frac{a}{q}+\lambda, b_i\right) &= \frac{1}{\phi(k)\phi(q)} C(\chi_0, q, 1, b_i, a) T(\lambda) \\ &\quad + \frac{1}{\phi(k)\phi(q)} \sum_{\xi \bmod k} \bar{\xi}(b_i) \\ &\quad \times \sum_{\chi \bmod q} C(\overline{\chi}, q, 1, b_i, a) W(\lambda, \xi\chi) + O(L^2). \end{aligned}$$

In the sequel, we will neglect the error term $O(L^2)$. We will see that its contribution will be dominated by other, larger error terms. For a fixed k, we introduce the following abbreviation:

$$R(N) = \int_{E_1(k)} S(\alpha,k,b_1)S(\alpha,k,b_2)S(\alpha,k,b_3)e(\alpha N), d\alpha. \tag{7.1}$$

Then,

$$R(N) = R^m(N) + R^e(N), \tag{7.2}$$

where

$$R^m(N) = \sum_{\substack{q \le P_1 \\ k \nmid q}} \frac{1}{\phi^3(k)\phi^3(q)} \sideset{}{^*}\sum_{a=1}^{q} \prod_{i=1}^{3} C(\chi_0, q, 1, b_i, a) \times e\Big(-\frac{a}{q}N\Big) \int_{-1/qQ}^{1/qQ} T^3(\lambda) e(-N\lambda)\, d\lambda,$$

(7.3)

$$\begin{aligned} R^e(N) = &\sum_{\substack{q \le P_1 \\ k \nmid q}} \frac{1}{\phi^3(k)\phi^3(q)} \sideset{}{^*}\sum_{a=1}^{q} e\Big(-\frac{a}{q}N\Big) \\ &\times \int_{-1/qQ}^{1/qQ} \prod_{i=1}^{3} \Big(\sum_{\xi \bmod k} \bar{\xi}(b_i) \sum_{\chi \bmod q} C(\overline{\chi}, q, 1, b_i, a) W(\lambda, \xi\chi) \Big) \\ &\times e\left(-\lambda N\right)\, d\lambda \\ &+ \sum_{i=1}^{3} \sum_{\substack{q \le P_1 \\ k \nmid q}} \frac{1}{\phi^3(k)\phi^3(q)} \sideset{}{^*}\sum_{a=1}^{q} e\Big(-\frac{a}{q}N\Big) \\ &\times \int_{-1/qQ}^{1/qQ} \prod_{\substack{j=1 \\ j \ne i}}^{3} \Big(\sum_{\xi \bmod k} \bar{\xi}(b_j) \sum_{\chi \bmod q} C(\overline{\chi}, q, 1, b_j, a) \\ &\times W(\lambda, \xi\chi) \Big) C(\chi_0, q, 1, b_i, a) T(\lambda) e\left(-\lambda N\right)\, d\lambda \\ &+ \sum_{i=1}^{3} \sum_{\substack{q \le P_1 \\ k \nmid q}} \frac{1}{\phi^3(k)\phi^3(q)} \sideset{}{^*}\sum_{a=1}^{q} e\Big(-\frac{a}{q}N\Big) \\ &\times \int_{-1/qQ}^{1/qQ} \Big(\sum_{\xi \bmod k} \bar{\xi}(b_i) \sum_{\chi \bmod q} C(\overline{\chi}, q, 1, b_i, a) W(\lambda, \xi\chi) \Big) \\ &\times \prod_{\substack{j=1 \\ j \ne i}}^{3} C(\chi_0, q, 1, b_j, a) T^2(\lambda) e\left(-\lambda N\right)\, d\lambda =: \sum_1 + \sum_2 + \sum_3 . \end{aligned}$$

Arguing exactly as in [**5**], we find

$$R^m(N) = \frac{1}{\phi^3(k)} \sum_{\substack{q \le P_1 \\ k \nmid q}} A(N,q,1) \frac{N^2}{32} + O\left(N^2 k^{-4} L^{-A}\right). \tag{7.4}$$

In the sequel we will without further mention use the fact that for any character χ induced by a primitive character χ^*, we have $W(\lambda, \chi) = W(\lambda, \chi^*) + O(L^2)$. Using Lemma 3.3, we estimate $\sum_1$:

(7.5)

$$\begin{aligned}
\left| \sum_1 \right| \le \frac{1}{\phi^3(k)} & \sum_{\substack{q \le P_1 \\ k \nmid q}} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} \sum_{\chi_3 \bmod q} \sum_{\xi_1 \bmod k} \sum_{\xi_2 \bmod k} \sum_{\xi_3 \bmod k} \\
& \times |Z(N,q,1,\chi_1,\chi_2,\chi_3)| \int_{-1/qQ}^{1/qQ} \prod_{j=1}^{3} |W(\lambda, \chi_j \xi_j)| \, d\lambda \\
\le \frac{1}{\phi^3(k)} & \sum_{\substack{r_1 \le P_1 \\ k \nmid r_1}} \sum_{\substack{r_2 \le P_1 \\ k \nmid r_2}} \sum_{\substack{r_3 \le P_1 \\ k \nmid r_3}} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \\
& \times \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \sum_{\xi_1 \bmod k} \sum_{\xi_2 \bmod k} \sum_{\xi_3 \bmod k} \\
& \times \int_{-1/[r_1,r_2,r_3]Q}^{1/[r_1,r_2,r_3]Q} \prod_{j=1}^{3} \left(|W(\lambda, \chi_j \xi_j)| + L^2 \right) \, d\lambda \\
& \times \sum_{\substack{q \le P_1 \\ [r_1,r_2,r_3] | q}} |Z(N,q,1,\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0)| \\
\ll \frac{L}{\phi^3(k)} & \sum_{\substack{r_1 \le P_1 \\ k \nmid r_1}} \sum_{\substack{r_2 \le P_1 \\ k \nmid r_2}} \sum_{\substack{r_3 \le P_1 \\ k \nmid r_3}} [r_1, r_2, r_3]^{-1/2} \\
& \times \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \\
& \times \sum_{\xi_1 \bmod k} \sum_{\xi_2 \bmod k} \sum_{\xi_3 \bmod k} \\
& \times \int_{-1/[r_1,r_2,r_3]Q}^{1/[r_1,r_2,r_3]Q} \prod_{j=1}^{3} \left(|W(\lambda, \chi_j \xi_j)| + L^2 \right) \, d\lambda.
\end{aligned}$$

In the following, we will neglect the error terms L^2 in the last integral in (7.5) as their contribution will be dominated by other terms. As a character ξ modulo k is either primitive or the principal character modulo k, the following relation holds for all characters χ_i and ξ_i, $i \in \{1,2,3\}$, over which is summed in (7.5):

$$(\chi\xi)^* = \begin{cases} \chi^* & \text{if } \xi = \xi_0, \\ \chi^*\xi & \text{otherwise.} \end{cases} \tag{7.6}$$

Thus, we see from (7.5) and (7.6) and by the notation for s_i introduced in (2.2),

(7.7)

$$\begin{aligned} \sum_1 \ll k^{-3}L^2 \Bigg(& \sum_{\substack{r_1 \le P_1 k \\ k || r_1}} \sum_{\substack{r_2 \le P_1 k \\ k || r_2}} \sum_{\substack{r_3 \le P_1 k \\ k || r_3}} + \sum_{\substack{r_1 \le P_1 k \\ k || r_1}} \sum_{\substack{r_2 \le P_1 k \\ k || r_2}} \sum_{\substack{r_3 \le P_1 \\ k \nmid r_3}} \\ & + \sum_{\substack{r_1 \le P_1 k \\ k || r_1}} \sum_{\substack{r_2 \le P_1 \\ k \nmid r_2}} \sum_{\substack{r_3 \le P_1 \\ k \nmid r_3}} + \sum_{\substack{r_1 \le P_1 \\ k \nmid r_1}} \sum_{\substack{r_2 \le P_1 \\ k \nmid r_2}} \sum_{\substack{r_3 \le P_1 \\ k \nmid r_3}} \Bigg) [s_1, s_2, s_3]^{-1/2} \\ & \times \sum \chi_1 \bmod r_1{}^* \sum_{\chi_2 \bmod r_2}{}^* \sum_{\chi_3 \bmod r_3}{}^* \\ & \times \int_{-1/[s_1,s_2,s_3]Q}^{1/[s_1,s_2,s_3]Q} \prod_{j=1}^{3} |W(\lambda, \chi_j)| \, d\lambda =: \sum_{i=1}^{4} \sum_{1,i}, \end{aligned}$$

where each $\sum_{1,i}$ stands for one of the multiple sums in (7.7). For the sum $\sum_{1,1}$, we see

(7.8)

$$\begin{aligned} \sum_{1,1} \ll k^{-2}k^{-1/9} & \sum_{\substack{r_1 \le P_1 k \\ k || r_1}} \sum_{\chi_1 \bmod r_1}{}^* \max_{|\lambda| \le 1/s_1 Q} |W(\lambda, \chi_1)| \\ & \times k^{-4/9} \sum_{\substack{r_2 \le P_1 k \\ k || r_2}} \sum_{\chi_2 \bmod r_2}{}^* \left(\int_{-1/s_2 Q}^{1/s_2 Q} |W(\lambda, \chi_2)|^2 \, d\lambda \right)^{1/2} \\ & \times k^{-4/9} \sum_{\substack{r_3 \le P_1 k \\ k || r_3}} [s_1, s_2, s_3]^{-1/2} \\ & \times \sum_{\chi_3 \bmod r_3}{}^* \left(\int_{-1/s_3 Q}^{1/s_3 Q} |W(\lambda, \chi_3)|^2 \, d\lambda \right)^{1/2}. \end{aligned}$$

Since $s = [s_1, s_2, s_3] = [[s_1, s_2], s_3]$, where $s_i = r_i/k$, we use Lemma 4.1 for $F = A$ to estimate the sum over r_3 in (7.8) is

$$= k^{-4/9} \sum_{\substack{r_3 \le P_1 k \\ k || r_3}} [[s_1, s_2], s_3]^{-1/2} \sum_{\chi_3 \bmod r_3}{}^{*} \left(\int_{-1/s_3 Q}^{1/s_3 Q} |W(\lambda, \chi_3)|^2 \, d\lambda \right)^{1/2}$$
$$\le K^A[s_1, s_2] \ll [s_1, s_2]^{-1/2+\varepsilon} N^{1/2} L^c.$$

Applying again Lemma 4.1 for $F = A$, we see that the contribution of this quantity to the sum over r_2 is:

$$\ll N^{1/2} L^c k^{-4/9} \sum_{\substack{r_2 \le P_1 k \\ k || r_2}} [s_1, s_2]^{-1/2+\varepsilon} \sum_{\chi_2 \bmod r_2}{}^{*} \left(\int_{-1/s_2 Q}^{1/s_2 Q} |W(\lambda, \chi_2)|^2 \, d\lambda \right)^{1/2}$$
$$= K^A[s_1] \ll s_1^{-1/2+\varepsilon} N L^c.$$

Inserting the last bound into (7.8) and using Lemma 4.2 for $F = A$, we see

$$\begin{aligned} \sum_{1,1} &\ll k^{-2} k^{-1/9} N \sum_{\substack{r_1 \le P_1 k \\ k || r_1}} s_1^{-1/2+\varepsilon} \sum_{\chi_1 \bmod r_1}{}^{*} \max_{|\lambda| \le 1/s_1 Q} |W(\lambda, \chi_1)| \\ &= k^{-2} N L^c J^A \ll k^{-2} N^2 L^{-A}. \end{aligned} \tag{7.9}$$

Arguing as in (7.8), we estimate the sum $\sum_{1,4}$ as follows:

$$\begin{aligned} \sum_{1,4} \ll k^{-2} L^2 k^{-1/9} &\sum_{r_1 \le P_1} \sum_{\chi_1 \bmod r_1}{}^{*} \max_{|\lambda| \le 1/r_1 Q} |W(\lambda, \chi_1)| \\ &\times k^{-4/9} \sum_{r_2 \le P_1} \sum_{\chi_2 \bmod r_2}{}^{*} \left(\int_{-1/r_2 Q}^{1/r_2 Q} |W(\lambda, \chi_2)|^2 \, d\lambda \right)^{1/2} \\ &\times k^{-4/9} \sum_{r_3 \le P_1} [r_1, r_2, r_3]^{-1/2} \sum_{\chi_3 \bmod r_3}{}^{*} \left(\int_{-1/r_3 Q}^{1/r_3 Q} |W(\lambda, \chi_3)|^2 d\,\lambda \right)^{1/2}. \end{aligned}$$

Now applying Lemmas 4.1 and 4.2 iteratively for $F = B$ in the same way as for $F = A$ in (7.8)–(7.9), we see that

$$\sum_{1,4} \ll k^{-2} N^2 L^{-A}. \tag{7.10}$$

Applying Lemmas 4.1 and 4.2 for $F = A$ and $F = B$, respectively, we obtain in the same way

$$\sum_{i=2}^{3} \sum_{1,i} \ll k^{-2} N^2 L^{-A}. \tag{7.11}$$

We note that

$$\max_{|\lambda| \le 1/Q} |T(\lambda)| \ll N, \tag{7.12}$$

and using $T(\lambda) \le \min(N, 1/\lambda)$, we see that

$$\left(\int_{-1/Q}^{1/Q} |T(\lambda)|^2 \, d\lambda \right)^{1/2} \ll N^{1/2}. \tag{7.13}$$

We apply these estimates and the arguments above to show

$$\sum_{2} + \sum_{3} \ll k^{-2} N^2 L^{-A}. \tag{7.14}$$

Therefore, we see from (7.3), (7.7), (7.9)–(7.11) and (7.14):

$$R^e(N) \ll k^{-2} N^2 L^{-A}. \tag{7.15}$$

In summary, we obtain from (7.2), (7.4) and (7.15),

$$R(N) = \frac{1}{\phi^3(k)} \sum_{\substack{q \le P_1 \\ k \nmid q}} A(N, q, 1) \frac{N^2}{32} + O\left(N^2 k^{-2} L^{-A}\right). \tag{7.16}$$

Now, in view of (7.1), the estimate (2.7) follows directly from (7.16).

7.2. The contribution of the interval over $E_2(k)$. In order to estimate the integral over $E_2(k)$, we closely follow an approach from [**11**, Section 3]. We first quote the following lemma from [**11**]:

Lemma 7.1. *Let $Z \ge 1$ be arbitrary. For any $A > 0$, there exists $D = D(A) > 0$ such that if $1 \le K \le Z^{2/3} N^{1/3} L^{-D}$, $\theta = Z^2 K^{-3} L^{-D}$, then*

$$\sum_{q \le K} \max_{|\lambda| \le \theta} \frac{1}{\phi(q)} \sum_{\chi \bmod q} |\tau(\overline{\chi})| \left| \sum_{N/4 < n \le N} \chi(n) \Lambda(n, \chi) e(n\lambda) \right| \ll Z N L^{-A}.$$

Here and in the sequel, $\tau(\chi)$ *is defined as* $\tau(\chi) = \sum_{a=1}^{q} \chi(a)e(a/q)$, *and* $\Lambda(n\chi) = \Lambda(n) - 1$ *if* $\chi = \chi_0$, *and* $\Lambda(n\chi) = \Lambda(n)$ *otherwise.*

Following the argument in [**11**], we now apply Lemma 7.1 to prove

Lemma 7.2. *For* $k|q$, *we define*

$$f(k,b,a,b) = \begin{cases} (\mu(q_1)/\phi(q))e(ab\overline{q_1}/k) \\ \qquad\qquad \text{if } k||q,\ q = q_1 k,\ q_1\overline{q_1} \equiv 1 \pmod{k}, \\ 0 \\ \qquad\qquad k^2|q. \end{cases}$$

$$\begin{aligned} &F(k,q,a,b,\lambda) \\ &\quad = \sum_{\substack{N/4<n\le N \\ n\equiv \pmod{k}}} \Lambda(n)e\left(n\left(\frac{a}{q}+\lambda\right)\right) - f(k,q,a,b) \sum_{N/4<n\le N} e(n\lambda), \end{aligned}$$

$$E(k,q) = \max_{(a,q)=1} \max_{(b,k)=1} \max_{|\lambda|\le 1/(qQ)} |F(k,q,a,b,\lambda)|.$$

For $R \le N^{1/6}L^{-E}$, P_1, P_2, Q *defined as in* (2.5) *and for any* $A > 0$ *and any sufficiently large* $E = E(A)$, *there is:*

$$\sum_{k\sim R} \sum_{\substack{q\le P_2 \\ k|q}} E(k,q) \ll NL^{-A}. \tag{7.17}$$

Proof. Arguing exactly as in [**11**, the proof of Lemma 3.2, Cases 2 and 3, equation (3.7)], we see that the lefthand side of (7.17) is upper bounded as follows:

$$\begin{aligned} &\ll \sum_{q\le P_2} \max_{|\lambda|\le 1/qQ} \frac{1}{\phi(q)} \sum_{\chi \bmod q} |\tau(\overline{\chi})| \left| \sum_{N/4<n\le N} \chi(n)\Lambda(n,\chi)e(n\chi) \right| \\ &\quad + O(P_2L^2) \\ &\ll L \max_{U} \sum_{q\sim U} \max_{|\lambda|\le 1/UQ} \frac{1}{\phi(q)} \sum_{\chi \bmod q} |\tau(\overline{\chi})| \left| \sum_{N/4<n\le N} \chi(n)\Lambda(n,\chi)e(n\chi) \right| \\ &\quad + O(P_2L^2) \end{aligned}$$

where the maximum over U is taken over all

$$U \le P_2 = k^2 L^{3G} \le N^{1/3} L^{-D}, \qquad \frac{1}{UQ} \le U^{-3} L^{-D}. \tag{7.18}$$

Choosing $Z = 1$ in Lemma 7.1, in view of the definitions of Q and U, see (2.5) and (7.18), we see that the optimal choice of k is $k = N^{1/6} L^{-E}$.

We now use Lemma 7.2 to establish the estimate (2.8). We know from [**11**, Section 4] that

$$\sum_{k\sim R} k \left| \int_{E_2(k)} S(\alpha,k,b_1)S(\alpha,k,b_2)S(\alpha,k,b_3)e(-\alpha N)\,d\alpha - \sum_{\substack{q\le P_2\\ k|q}} A(N,q,k)\frac{N^2}{32} \right|$$
$$\ll \sum_{k\sim R}\left[\frac{N^2L}{kP_2} + \frac{P_2Q^2}{k} + \frac{N}{k}\sum_{\substack{q\le P_2/k\\ k||q}} E(k,q) + \frac{N}{k}\sum_{\substack{q\le P_2/k^2\\ k^2||q}} E(k,q)\right].$$

Applying the estimate (7.17) to (7.19), we obtain the estimate (2.8). □

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