Acta Mathematica Sinica, English Series
Aug., 2005, Vol. 21, No. 4, pp. 833–840
Published online: June 19, 2005
DOI: 10.1007/s10114-004-0506-0
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Sums of Five Almost Equal Prime Squares

Claus BAUER
Dolby Laboratories, 100 Potrero Avenue, San Francisco, CA, 94103, USA
E-mail: cb@dolby.com

Abstract Let $P_i, 1 \leq i \leq 5$, be prime numbers. It is proved that every integer N that satisfies $N \equiv 5 \,(\mathrm{mod}\, 24)$ can be written as $N = p_1^2 + p_2^2 + p_3^2 + P_4^2 + p_5^2$, where $\left|\sqrt{N}5 - p_i\right| \leq N^{\frac{1}{2} - \frac{19}{850} + \epsilon}$.

Keywords Prime numbers, Exponential sums, L-functions
MR(2000) Subject Classification 11P32, 11L07

1 Introduction

Among numerous results, Hua [1] proved that every sufficiently large integer satisfying $n \equiv 5 \,(\mathrm{mod}\, 24)$ is equal to the sum of five prime squares. Liu and Zhan [2] could improve this result by proving the following:

Theorem 1 *Assume the Great Riemann Hypothesis. Then any sufficiently large integer n satisfying $n \equiv 5 \,(\mathrm{mod}\, 24)$ can be written as*

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2, \tag{1.1}$$

where $\left|p_i - \sqrt{\frac{n}{5}}\right| \leq y,\ i = 1,\ 2,\ 3,\ 4,\ 5$ *for* $y = n^{\frac{9}{20}+\epsilon}$.

In [3] the same problem was investigated without assuming the Great Riemann Hypothesis. It was proved that (1.1) holds for

$$y = n^{\frac{1}{2}-\delta}, \tag{1.2}$$

for a $\delta \geq 0$. The proof uses the ideas of Liu and Tsang ([4, 5]). The exact value of δ depends on the existence of the Siegel zero of the Dirichlet series and is not exactly calculated. Liu and Zhan ([6]) could further improve on this result by showing that (1.2) holds for $\delta = \frac{1}{50} - \epsilon$, $\forall \epsilon > 0$. This result gives not only a fixed value for δ, but also a value for δ that does not depend on the existence of the possible Siegel zero of the Dirichlet series. Here we will further improve on this result by proving the following theorem:

Theorem *Any sufficiently large positive integer n satisfying $n \equiv 5 \,(\mathrm{mod}\, 24)$ can be written as*

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2, \tag{1.3}$$

where $\left|p_i - \sqrt{\frac{n}{5}}\right| \leq y,\ i = 1,\ 2,\ 3,\ 4,\ 5$ *for* $y = n^{\frac{1}{2}-\frac{19}{850}+\epsilon}$.

2 Preliminaries and Outline of the Proof

(a, b) and $[a, b]$ denote the greatest common divisor and the smallest common multiple of two integers a and b, respectively. Let $L = \log x$, $e(x) = e^{2\pi i x}$, $N_1 = \sqrt{\frac{n}{5}} - y$, $N_2 = \sqrt{\frac{n}{5}} + y$,

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} = \sum_{a=1}^{q}{}^{*}, \quad \sum_{\substack{\chi \bmod q\\ \chi \text{ primitive}}} = \sum_{\chi \bmod q}{}^{*}, \quad S(\alpha) = \sum_{N_1 \leq m \leq N_2} \Lambda(m) e(m^2 \alpha),$$

Received September 10, 2003, Accepted September 19, 2003

$$R(n)=\sum_{\substack{n=m_1^2+m_2^2+m_3^2+m_4^2+m_5^2\\ N_1<m_i\le N_2}}\Lambda(m_1)\Lambda(m_2)\Lambda(m_3)\Lambda(m_4)\Lambda(m_5).$$

Define for a character $\chi \bmod q$ $C(a,\chi)=\sum_{h=1}^{q}\chi(h)e(\frac{a}{q}h^2)$, $C(a,\chi_0)=C(a,q)$. Let c and $\epsilon,\epsilon_1,...>0$ denote constants that may take different values on different occasions. We shall write $x^\epsilon L^c\ll x^\epsilon$, $x^{\epsilon_1}x^{\epsilon_1}\ll x^{\epsilon_1}$. Set

$$P=n^{2+\epsilon_1}y^{-4},\quad Q=y^7n^{-\frac{5}{2}-2\epsilon_1}. \tag{2.1}$$

We define the major arcs M and the minor arcs m by $M=\cup_{q\le P}\cup^{q}_{\substack{a=1\\(a,q)=1}}[\frac{a}{q}-\frac{1}{Qq},\ \frac{a}{q}+\frac{1}{Qq}]$, $m=[-\frac{1}{Q},1-\frac{1}{Q}]\setminus M$. We have

$$R(n)=\int_M S^5(\alpha)e(-n\alpha)\,d\alpha\ +\int_m S^5(\alpha)e(-n\alpha)\,d\alpha=:R_1(n)+R_2(n). \tag{2.2}$$

We will prove that $R(n)>0$ for sufficiently large n that satisfy the congruence conditions in (1.1). This proves Theorem 2.

For the treatment of the minor arcs we quote the following lemma due to Harman [7]:

Lemma 2.1 *Suppose $\epsilon>0$ is given and $|q\alpha-a|<q^{-1}$ with $(a,q)=1$. Then*

$$\sum_{x\le n\le x+y}\Lambda(n)\,e(n^2\alpha)\ll y^{1+\epsilon}\left(\frac{1}{q}+\frac{x^{\frac12}}{y}+\frac{x^{\frac43}}{y^2}+\frac{qx}{y^3}\right)^{\frac14}$$

holds for $1\le q\le xy$.

Applying this to $S(\alpha)$ we find that

$$\max_{\alpha\in m}|S(\alpha)|\ll y^{1+\epsilon}\left(P^{-1/4}+\frac{n^{\frac{1}{16}}}{y^{\frac14}}+\frac{n^{\frac16}}{y^{\frac12}}+\frac{Q^{\frac14}n^{\frac18}}{y^{\frac34}}\right)\ll y^2n^{-\frac12-\epsilon_1/8}, \tag{2.3}$$

by choosing $\epsilon_1\ge 8\epsilon$. Using (2.3) we estimate the contribution of the minor arcs as

$$R_2(n)\le\sup_{\alpha\in m}|S(\alpha)|\int_0^1|S(\alpha)|^4\,d\alpha\ll y^4n^{-\frac12}L^{-B}, \tag{2.4}$$

for any $B>0$. In the following sections we shall first show that, for any $B>0$

$$R_1(n)=\frac{1}{32}P_0\sum_{q\le P}\frac{Y(q)}{\phi^5(q)}+O(y^4x^{-\frac12}L^{-B}), \tag{2.5}$$

where

$$y^4x^{-1/2}\ll P_0=\sum_{\substack{m_1+m_2+m_3+m_4+m_5=n\\ N_1^2<m_i\le N_2^2}}\frac{1}{\sqrt{m_1m_2m_3m_4m_5}}\ll y^4x^{-1/2}, \tag{2.6}$$

if $n\in]x/2,x]$. We further define

$$Z(q,\chi_1,\chi_2,\chi_3,\chi_4,\chi_5)=\sum_{a=1}^{q}{}^{*}C(a,\chi_1)C(a,\chi_2)C(a,\chi_3)C(a,\chi_4)C(a,\chi_5)e\left(-\frac{a}{q}n\right),$$

$$Z(q,\chi_0,\chi_0,\chi_0,\chi_0,\chi_0)=Y(q),\quad A(q)=\frac{Y(q)}{\phi^5(q)},\quad s(p)=\left\{\begin{matrix}1+A(p), & p>2,\\ 1+A(2)+A(4)+A(8), & p=2.\end{matrix}\right\}.$$

Finally we will derive

$$R_1(n)=\frac{1}{32}P_0\prod_{p\ge1}s(p)+O(y^4x^{-1/2}L^{-B}), \tag{2.7}$$

where $\prod_{p\ge1}s(p)>c$, from (2.5). The theorem follows from (2.2), (2.4), (2.6) and (2.7).

3 Treatment of the Major Arcs

We define

$$S(\lambda,\chi)=\sum_{N_1<m\le N_2}\Lambda(m)\chi(m)e(m^2\lambda),\quad T(\lambda)=\sum_{N_1<m\le N_2}e(m^2\lambda),$$

$$W(\lambda,\chi)=S(\lambda,\chi)-E_0T(\lambda),\quad E_0=\left\{\begin{array}{ll}1, & \text{if}\,\chi=\chi_0,\\ 0, & \text{otherwise.}\end{array}\right\}.$$

In the following we will appeal to the following lemma which is contained in Lemmas 5.1 and 5.2 in [8]:

Lemma 3.1 *If $(a,q)=1$, then $C(a,\chi)\ll q^{1/2+\epsilon}$.*

Splitting the summation over m in the rest of the classes modulo q we obtain

$$S\left(\frac{a}{q}+\lambda\right)=\frac{C(a,q)}{\phi(q)}T(\lambda)+\frac{1}{\phi(q)}\sum_{\chi \bmod q}C(a,\chi)W(\lambda,\chi)+O(L^2).$$

Thus we derive from (2.2) that

$$R_1(n)=R_1^m(n)+R_1^e(n)+O\left(x^{\frac{5}{2}+3\epsilon_1}y^{-3}\right), \tag{3.1}$$

where

$$R_1^m(n)=\sum_{q\le P}\frac{1}{\phi^5(q)}\sum_{a=1}^{q}{}^{*}C^5(a,q)e\left(-\frac{a}{q}n\right)\int_{-1/Qq}^{1/Qq}T^5(\lambda)e(-n\lambda)d\,\lambda,$$

$$\begin{aligned}R_1^e(n)=&\sum_{q\le P}\frac{1}{\phi^5(q)}\sum_{a=1}^{q}{}^{*}\int_{-1/Qq}^{1/Qq}\Big(\sum_{\chi}C(a,\chi)W(\lambda,\chi)\Big)^5e\Big(-\frac{a}{q}n-\lambda n\Big)d\,\lambda\\
&+5\sum_{q\le P}\frac{1}{\phi^5(q)}\sum_{a=1}^{q}{}^{*}\int_{-1/Qq}^{1/Qq}C(a,q)T(\lambda)\Big(\sum_{\chi}C(a,\chi)W(\lambda,\chi)\Big)^4e\left(-\frac{a}{q}n-\lambda n\right)d\,\lambda\\
&+10\sum_{q\le P}\frac{1}{\phi^5(q)}\sum_{a=1}^{q}{}^{*}\int_{-1/Qq}^{1/Qq}(C(a,q)T(\lambda))^2\left(\sum_{\chi}C(a,\chi)W(\lambda,\chi)\right)^3e\left(-\frac{a}{q}n-\lambda n\right)d\,\lambda\\
&+10\sum_{q\le P}\frac{1}{\phi^5(q)}\sum_{a=1}^{q}{}^{*}\int_{-1/Qq}^{1/Qq}(C(a,q)T(\lambda))^3\left(\sum_{\chi}C(a,\chi)W(\lambda,\chi)\right)^2e\left(-\frac{a}{q}n-\lambda n\right)d\,\lambda\\
&+5\sum_{q\le P}\frac{1}{\phi^5(q)}\sum_{a=1}^{q}{}^{*}\int_{-1/Qq}^{1/Qq}(C(a,q)T(\lambda))^4\sum_{\chi}C(a,\chi)W(\lambda,\chi)e\left(-\frac{a}{q}n-\lambda n\right)d\,\lambda\\
=:&\sum_1+5\sum_2+10\sum_3+10\sum_4+5\sum_5.\end{aligned} \tag{3.2}$$

We first evaluate the main term R_1^m. We will use the following lemmas:

Lemma 3.2 *Let $f(x)$, $g(x)$ be monotonic functions in the interval $[a,b]$ and $|g(x)|\ll M$.*

(i) *If $|f'(x)|\gg m>0$, then $\int_a^b g(x)\,e(f(x))\,dx\ll M/m$.*

(ii) *If $|f''(x)|\gg r>0$, then $\int_a^b g(x)\,e(f(x))\,dx\ll M/r^{\frac{1}{2}}$.*

(iii) *If $|f'(x)|\le\theta<1$, $g(x),g'(x)\ll 1$, $\sum_{a<n\le b}g(n)e(f(n))=\int_a^b g(x)\,e(f(x))\,dx+O(\frac{1}{1-\theta})$.*

Proof See Lemma 4.8 in [9] and Chapter 21 in [10].

Lemma 3.3 $\frac{|Z(q,\chi_0\chi_1,\chi_0\chi_2,\chi_0\chi_3,\chi_0\chi_4,\chi_0\chi_5)|}{\phi^5(q)}\ll r^{-3/2+\epsilon}(\log P)^c$.

Proof Let I denote the left-hand side in Lemma 3.3 and write $Z(q)=Z(q,\chi_0\chi_1,\chi_0\chi_2,\chi_0\chi_3)$. Arguing as in Lemma 6.7, [11] we obtain $I\ll\sum_{u|a}\frac{|Z(ur)|}{\phi^5(ur)}\sum_{\substack{q\le Q/ur\\(q,r)=1}}|A(q)|$, where $a\ll 1$. Using Lemma 3.1 we find that $\sum_{u|a}\frac{|Z(ur)|}{\phi^5(ur)}\ll r^{-3/2+\epsilon}$. Thus Lemma 3.3 follows from

$$\sum_{q\le P}|A(q)|\ll(\log P)^c. \tag{3.3}$$

To prove (3.3) we argue as in Lemma 5.4 a) and the proof of Lemma 6.3 c) in [11] and get

$$\sum_{q\le P}|A(q)| \ll \prod_{p\le P}\left(1+\frac{c}{p}\right) \ll (\log\, P)^c.$$

Now we apply Lemma 3.2 to $T(\lambda)$ and find

$$T(\lambda)=\int_{N_1}^{N_2} e(\lambda u^2)d\,u+O(1)=\frac{1}{2}\int_{N_1^2}^{N_2^2} v^{-1/2}e(\lambda v)d\,v+O(1)=\frac{1}{2}\sum_{N_1^2<m\le N_2^2}\frac{e(\lambda m)}{\sqrt{m}}+O(1).$$

Substituting this in $R_1^m(n)$ we see

$$\begin{aligned} R_1^m(n) &= \frac{1}{32}\sum_{q\le P}\frac{Y(q)}{\phi^5(q)}\int_{-1/Qq}^{1/Qq}\Bigg(\sum_{N_1^2<m\le N_2^2}\frac{e(\lambda m)}{\sqrt{m}}\Bigg)^5 e(-n\lambda)d\,\lambda \\ &\quad +O\Bigg(\sum_{q\le P}\frac{|Y(q)|}{\phi^5(q)}\int_{-1/Qq}^{1/Qq}\Bigg|\sum_{N_1^2<m\le N_2^2}\frac{e(\lambda m)}{\sqrt{m}}\Bigg|^4 d\,\lambda\Bigg). \end{aligned} \tag{3.4}$$

Using

$$\sum_{N_1^2<m\le N_2^2}\frac{e(\lambda m)}{\sqrt{m}} \ll \min\left(y,\, \frac{1}{\sqrt{x}|\lambda|}\right) \tag{3.5}$$

and Lemma 3.3 with $r=1$ we derive, from (3.4),

$$\begin{aligned} R_1^m(n) &= \frac{1}{32}\sum_{q\le P}\frac{Y(q)}{\phi^5(q)}\int_{-1/2}^{1/2}\Bigg(\sum_{N_1^2<m\le N_2^2}\frac{e(\lambda m)}{\sqrt{m}}\Bigg)^5 e(-n\lambda)d\,\lambda+O\left(y^4x^{-1/2}L^{-B}\right) \\ &\quad +O\Bigg(\sum_{q\le P}\left|\frac{Y(q)}{\phi^5(q)}\right|\int_{1/Qq}^{1/2}\frac{1}{(\sqrt{x}|\lambda|)^5}d\,\lambda\Bigg) \\ &= \frac{1}{32}P_0\sum_{q\le P}\frac{Y(q)}{\phi^5(q)}+O\left((PQ)^4x^{-5/2}\right)+O\left(y^4x^{-1/2}L^{-B}\right) \\ &= \frac{1}{32}P_0\sum_{q\le P}\frac{Y(q)}{\phi^5(q)}+O(y^4x^{-1/2}L^{-B}), \end{aligned} \tag{3.6}$$

$\forall\, B>0$, where P_0 is defined as in (2.6). Applying Lemma 3.3 we can estimate $\sum_1$ in the following way:

$$\begin{aligned} \Big|\sum_1\Big| &= \Bigg|\sum_{q\le P}\frac{1}{\phi^5(q)}\sum_{\chi_1 \bmod q}\sum_{\chi_2 \bmod q}\sum_{\chi_3 \bmod q}\sum_{\chi_4 \bmod q}\sum_{\chi_5 \bmod q} \\ &\qquad Z(q,\chi_1,\chi_2,\chi_3,\chi_4,\chi_5)\int_{-1/Qq}^{1/Qq}\prod_{j=1}^{5}W(\lambda,\chi_j)e(-n\lambda)d\lambda\Bigg| \\ &\le \sum_{r_1\le P}\sum_{r_2\le P}\sum_{r_3\le P}\sum_{r_4\le P}\sum_{\substack{r_5\le P\\ [r_1,r_2,r_3,r_4,r_5]\le P}}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\sideset{}{^*}\sum_{\chi_2 \bmod r_2} \\ &\qquad \sideset{}{^*}\sum_{\chi_3 \bmod r_3}\sideset{}{^*}\sum_{\chi_4 \bmod r_4}\sideset{}{^*}\sum_{\chi_5 \bmod r_5}\int_{-1/Q[r_1,r_2,r_3,r_4,r_5]}^{1/Q[r_1,r_2,r_3,r_4,r_5]}\prod_{j=1}^{5}|W(\lambda,\chi_j)|\,d \\ &\quad\times \sum_{\substack{q\le P\\ [r_1,r_2,r_3,r_4,r_5]|q}}\frac{|Z(q,\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0,\chi_4\chi_0,\chi_5\chi_0|}{\phi^5(q)} \end{aligned}$$

$$\ll L^c \sum_{r_1\le P}\sum_{r_2\le P}\sum_{r_3\le P}\sum_{r_4\le P}\sum_{r_5\le P}[r_1,r_2,r_3,r_4,r_5]^{-\frac{3}{2}+\epsilon}\sum_{\chi_1 \bmod r_1}{}^{*}\sum_{\chi_2 \bmod r_2}{}^{*}\sum_{\chi_3 \bmod r_3}{}^{*}$$

$$\sum_{\chi_4 \bmod r_4}{}^{*}\sum_{\chi_5 \bmod r_5}{}^{*}\int_{-1/Q[r_1,r_2,r_3,r_4,r_5]}^{1/Q[r_1,r_2,r_3,r_4,r_5]}\prod_{i=1}^{5}|W(\lambda,\chi_i)|d\lambda.$$

Using $[r_1,r_2,r_3,r_4,r_5]^{\frac{3}{2}} \ge (r_1r_2)^{\frac{39}{152}}(r_3r_4r_5)^{\frac{25}{76}}$ we obtain

$$\sum_1 \ll L^c \max_{|\lambda|\le 1/Q} I^3(\lambda)W^2, \tag{3.7}$$

where

$$I(\lambda)=\sum_{r\le P} r^{-25/76+\epsilon}\sum_{\chi}{}^{*}|W(\lambda,\chi)|, \quad W=\sum_{r\le P} r^{-39/152+\epsilon}\sum_{\chi}{}^{*}\left(\int_{-1/Qr}^{1/Qr}|W(\lambda,\chi)|^2 d\,\lambda\right)^{1/2}.$$

Arguing similarly we obtain

$$5\sum_2 +10\sum_3 +10\sum_4 +5\sum_5 \ll L^cW^2I^2T+W^2IT^2+W^2T^3+WT^3S, \tag{3.8}$$

where $T=\max_{|\lambda|\le 1/Q}|T(\lambda)| \ll y$, and using (3.5) we get $S=(\int_{-1/Q}^{1/Q}|T(\lambda)|^2\,d\lambda)^{1/2} \ll y^{1/2}x^{-1/4}$. Thus we see from (3.1), (3.2), (3.6)–(3.8) that the proof of (2.5) reduces to the proof of the following two lemmas:

Lemma 3.4 *If $P \le n^{\frac{38}{425}-\epsilon_2}$, then $W \ll_B y^{1/2}x^{-1/4}L^{-B}$ for any $B>0$.*

Lemma 3.5 *If $P \le n^{\frac{38}{425}-\epsilon_2}$, then $\max_{|\lambda|\le 1/Q} I(\lambda) \ll yL^A$ for a certain $A>0$.*

For the proof of these lemmas we will appeal to the following results:

Lemma 3.6 *For any $P\ge 1$, $T\ge 1$ and $k=0,1$*

$$\sum_{q\le P}\sum_{\chi \bmod q}{}^{*}\int_{-T}^{T}\left|L^{(k)}\left(\frac{1}{2}+it,\chi\right)\right|^4 dt \ll P^2T(\log PT)^{4(k+1)}.$$

Lemma 3.7 *For any $P\ge 1$, $T\ge 1$ and any complex numbers a_n*

$$\sum_{q\le P}\sum_{\chi \bmod q}{}^{*}\int_{-T}^{T}\left|\sum_{n=M+N}^{M}a_n\chi(n)n^{-it}\right|^2 dt \ll \sum_{n=M+N}^{M}(P^2T+n)|a_n|^2.$$

Lemma 3.8 *Let $N^*(\alpha,T,q)$ denote the number of zeros $\sigma+it$ of all L-functions to primitive characters modulo q within the region $\sigma\ge\alpha$, $|t|\le T$. Then*

$$\sum_{q\le Q}N^*(\alpha,T,q)\ll (Q^2T)^{12(1-\alpha)/5}(\log Q^2T)^c.$$

These three lemmas may be found in [8].

4 Proof of Lemma 3.4

Let

$$W=\sum_{R\le P}W_R, \tag{4.1}$$

with $W_R=\sum_{r\sim R}r^{-39/152+\epsilon}\sum_{\chi}{}^{*}(\int_{-1/Qr}^{1/Qr}|W(\lambda,\chi|^2)^{1/2}$. To prove the lemma it is enough to show that

$$W_R \ll y^{1/2}x^{-1/4}L^{-B-1}, \quad \forall\, B>0. \tag{4.2}$$

Applying Lemma 1, [12] and setting $X=\max\left(t,N_1^2\right)$ and $X+Y=\min\left(t+Qr,N_2^2\right)$, we get

$$\int_{-1/Qr}^{1/Qr}|W(\lambda,\chi)|^2\,d\,\lambda \ll (QR)^{-2}\int_{N_1^2-QR}^{N_2^2}\left|\sum_{X\le m^2\le X+Y}\Lambda(m)\chi(m)-E_0\right|^2 dt. \tag{4.3}$$

In the following we will treat the cases $R > L^D$ and $R \le L^D$ separately for a sufficiently large constant $D > 0$. In the first case we argue exactly as in part III, [13] and find that the inner sum in (4.3) is a linear combination of $O(L^c)$ terms of the form

$$S_{I_{a_1},..,I_{a_{2k+1}}} = \frac{1}{2\pi i}\int_{-T}^{T} F\left(\frac{1}{2}+it,\chi\right)\frac{(X+Y)^{\frac{1}{2}(\frac{1}{2}+iu)} - X^{\frac{1}{2}(\frac{1}{2}+iu)}}{\frac{1}{2}+iu}du \ + O(T^{-1}x^{\frac{1}{2}+\epsilon}),$$

where $2 \le T \le x$,

$$F(s,\chi) = \prod_{j=1}^{10} f_j(s,\chi), \quad f_j(s,\chi) = \sum_{n\in I_j} a_j(n)\chi_n n^{-s}, \quad a_j(n) = \left\{\begin{array}{ll} \log n \text{ or } 1, & j=1, \\ 1, & 1<j\le 10, \\ \mu(n), & 6\le n\le 10. \end{array}\right\},$$

$$\sqrt{x} \ll \prod_{j=1}^{10} N_j \ll \sqrt{x}, \quad N_j \le x^{1/10}, \quad 6\le j\le 10. \tag{4.4}$$

We see $\frac{(X+Y)^{\frac{1}{2}(\frac{1}{2}+iu)} - X^{\frac{1}{2}(\frac{1}{2}+iu)}}{\frac{1}{2}+iu}du \ll \min\left(QRx^{-3/4}, x^{1/4}(|u|+1)^{-1}\right)$. Taking $T = x^{2\epsilon}P^2$ and $T_0 = \frac{x}{QR}$ we derive from (4.3) that in order to prove (4.2) it is enough to show that

$$\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll x^{1/4}R^{39/152-\epsilon}L^{-B-1-c}, \quad R\le P, \tag{4.5}$$

$$\sum_{r\sim R}\sum_{\chi}{}^{*}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll T_1QR^{191/152-\epsilon}x^{-3/4}L^{-B-1-c}$$
$$\text{for } \ R\le P,\, T_0<|T_1|\le T. \tag{4.6}$$

For the proof of (4.5) and (4.6) we will first prove the following propositions:

Proposition 1 *If there exist* N_{j_1} *and* N_{j_2} $(1 \le j_1,\, j_2 \le 5)$ *such that* $N_{j_1}N_{j_2} \ge P^{85/38+\epsilon_3}$ *then* (4.5) *is true.*

Proof Without loss of generality we suppose that $j_1 = 1$, $a_1(n) = \log n$ and $j_2 = 2$, $a_2(n) = 1$. Arguing as in the proof of Proposition 1 in [13] and applying Lemma 3.6 we obtain

$$\begin{aligned}
&\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_0^{T_0}\left|f_1\left(\frac{1}{2}+it,\chi\right)\right|^4 dt \\
&\ll L^4\int_{-x^{1/2}}^{x^{1/2}}\frac{dv}{1+|v|}\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_v^{T_0+v}\left|L'\left(\frac{1}{2}+it,\chi\right)\right|^4 dt + T_0R^2L^4 \\
&\ll L^5\max_{|N|\le T_0}\int_{N/2}^{N}\frac{dv}{1+|v|}\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_v^{T_0+v}\left|L'\left(\frac{1}{2}+it,\chi\right)\right|^4 dt + T_0R^2L^4 \\
&\quad + L^5\max_{|N|\le x^{1/2}} N^{-1}\int_0^{T_0}dt\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_{\frac{N}{2}+t}^{N+t}\left|L'\left(\frac{1}{2}+iv,\chi\right)\right|^2 dv + T_0R^2L^4 \\
&\ll R^2T_0L^{10}.
\end{aligned}$$

Using Lemma 3.7 and the last result we find

$$\begin{aligned}
&\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \\
&\ll \left(\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|f_1\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\right)^{1/4}\left(\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|f_2\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\right)^{1/4}
\end{aligned}$$

$$\left(\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|\prod_{j=3}^{10}f_j\left(\frac{1}{2}+it,\chi\right)\right|^2dt\right)^{1/2}$$
$$\ll (R^2T_0)^{1/2}\left(R^2T_0+\frac{x^{1/2}}{N_{j1}N_{j2}}\right)^{1/2}L^c \ll x^{1/4}R^{\frac{39}{152}}L^{-B-1-c},$$

due to the choice of T_0 and P.

Proposition 2 *Let $J=\{1,...,10\}$. If J can be divided into two non-overlapping subsets J_1 and J_2 such that* $\max(\prod_{j\in J_1}N_j,\prod_{j\in J_2}N_j)\ll x^{\frac{1}{2}}P^{-\frac{85}{38}-\epsilon_4}$, *then (4.5) is true.*

Proof Let $F_i(s,\chi)=\prod_{j\in J_i}f_j(s,\chi)=\sum_{n\ll x^{1/2}P^{-\frac{85}{38}-\epsilon_4}}b_i(n)\chi(n)n^{-s}$, $b_i(n)\ll d_c(n)$, $i=1,2$, where $M_i=\prod_{j\in J_1}N_j$. Applying Lemma 3.7 and (4.4) we see

$$\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt$$
$$\ll\left(\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F_1\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/2}\left(\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F_2\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/2}$$
$$\ll (R^2T_0+M_1)^{1/2}(R^2T_0+M_2)^{1/2}\ll R^2T_0+x^{\frac{1}{4}}RP^{-\frac{85}{76}-\frac{\epsilon_4}{2}}T_0^{1/2}+x^{1/4}L^c.$$

This proves the lemma because of $R>L^D$. Now we can prove (4.5). In view of Proposition 1 and $P=x^{\frac{38}{425}-\epsilon_2}$ we assume $N_iN_j\le P^{85/38+\epsilon_5}\le x^{1/5}$, $1\le i,\ j\le 5,\ i\ne j$, from which we conclude that there is at most one $N_j\,(1\le j\le 10)$ satisfying $N_j\ge x^{1/10}$. Suppose this $N_j=N_{j_0}$, otherwise $N_{j0}=1$. Re-order the N_j: $N_{j_1}\ge N_{j_2}\ge\cdots\ge N_{j_k}$ ($k=9$ or 10). There is an integer $1\le l\le k-1$ such that $N_{j_0}N_{j_1}\cdots N_{j_{l-1}}\le x^{1/5}$ and $N_{j_0}N_{j_1}\cdots N_{j_l}\ge x^{1/5}$. Set $M_1=N_{j_0}N_{j_1}\cdots N_{j_l}$ and $M_2=N_{j_{l+1}}\cdots N_{j_k}$. We find $M_1\le x^{1/5}N_{j_l}\le x^{3/10}$ and $M_2\ll x^{1/2}M_1^{-1}\ll x^{3/10}$. The sets M_1 and M_2 fulfill the conditions of Proposition 2 and therefore (4.5) is proved. The proof of (4.6) goes along the same lines and is therefore omitted. (4.3) therefore holds in the case $q>L^D$. In the case $q\le L^D$ we can estimate the sum on the right-hand side of (4.3) by using the zero expansion of the von Mangoldt function:

$$\sum_{X\le m^2\le X+Y}\Lambda(m)\chi(m)-E_0\sum_{X\le m^2\le X+Y}1\ll\sum_{|\mathrm{Im}\,\rho|\le x^{1/6}}\left|\frac{(X+Y)^{\rho/2}}{\rho}-\frac{X^{\rho/2}}{\rho}\right|+O(x^{1/3}L^2)$$
$$\ll QRx^{-1/2}\sum_{|\mathrm{Im}\rho|\le x^{1/6}}x^{\frac{\beta-1}{2}}+O(x^{1/3}L^2),$$

where ρ runs over the non-trivial zeros of the L-function corresponding to χ with $|\mathrm{Im}\,p|\le x^{\delta}$ and $\beta=\mathrm{Im}\,\rho$. Now applying Lemma 3.8 and the fact that the L-functions to moduli $Q\le L^D$ have no zeros $\sigma+it$ in the region $\sigma\ge 1-\delta(T):1-\frac{c_0}{\log\,q+(\log(T+2))^{4/5}}$, $|t|\le T$, we choose $T=x^{1/6}$ and thus obtain, from (4.3),

$$\int_{-1/Qr}^{1/Qr}|W(\lambda,\chi)|^2\,d\lambda\ \ll\ yx^{-1/2}\left(\sum_{|\mathrm{Im}\,\rho|\le x^{1/6}}x^{\frac{\beta-1}{2}}\right)^2+(QR)^{-2}x^{\frac{7}{6}}yL^4$$
$$\ll yx^{-1/2}L^c\left(\max_{\frac{1}{2}\le\beta\le1-\delta(T)}x^{\frac{2}{5}(1-\beta)}x^{\frac{1}{2}(\beta-1)}\right)^2+x^{\frac{37}{6}+4\epsilon_1}y^{-13}L^c\ll yx^{-1/2}\exp(-cL^{1/5}),$$

for any $B>0$. This proves (4.1) for $R\le L^D$.

5 Proof of Lemma 3.5

To prove the lemma it is enough to show that $\max_{R\le P/2}\sum_{r\sim R}\sum_{\chi}{}^{*}|W(\lambda,\chi_r)|\ll yL^{A-1}R^{\frac{25}{76}-\epsilon}$, where $r \sim R$ denotes $R < r \leq 2R$. Arguing as in the section before we find that

$$W(\chi,\lambda) \ll L^c \max_{I_{a_1},..,I_{a_{2k+1}}} \left| \int_{-T}^{T} F\left(\frac{1}{2}+it,\chi\right) dt \int_{N_1^2}^{N_2^2} u^{-3/4} e\left(\frac{t}{4\pi}\log u + \lambda u\right) du \right| + yx^{-\epsilon}P^{-2},$$

if

$$T = x^{\frac{1}{2}+2\epsilon} y^{-1} P^2 (1+|\lambda|x). \tag{5.1}$$

Estimating the inner integral by Lemma 3.2 we obtain

$$\int_{N_1^2}^{N_2^2} u^{-3/4} \left(\frac{t}{4\pi}\log u + \lambda u\right) du| \ll x^{-3/4} \min\left(yx^{1/2}, \frac{x}{\sqrt{|t|+1}}, \frac{x}{\min_{N_1<u\leq N_2} |t+4\pi\lambda u|}\right).$$

Taking and $T_0 = xy^{-2}$ and $T_1 = 1 + 8\pi|\lambda|u$ we conclude that in order to prove the lemma it is enough to prove that

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{0}^{T_0} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll x^{1/4} R^{\frac{25}{76}-\epsilon} L^c, \quad R \leq P/2,$$

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{T_0}^{T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll yx^{-1/4} T_1^{\frac{1}{2}} R^{\frac{25}{76}-\epsilon} L^c, \quad R \leq P/2,$$

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll yx^{-1/4} T_1 R^{\frac{25}{76}-\epsilon} L^c, \quad T_0 \leq |T_1| \leq 2T,\ R \leq P/2.$$

These estimates can be shown in the same way as the estimates (4.5) and (4.6). Because of $A > 0$ the proof works here for all $q \geq 1$.

6 Proof of Theorem 1

We now derive (2.7) from (2.5). We use

Lemma 6.1 $\sum_{q\leq P} A(q) = \prod_{p\leq P} s(p) + O\left(P^{-1/2+\epsilon}\right)$, *where* $\prod_{p\leq P} s(p) > c > 0$.

Proof This is Lemma 4.2 in [2]. Applying Lemma 6.1 to (2.5) yields (2.7).

References

[1] Hua, L. K.: Some results in the additive prime number theory. *Quart. J. Math.*, **9**, 68–80 (1938)

[2] Liu, J. Y., Zhan, T.: On sums of five almost equal prime squares. *Acta Arith.*, **77**, 369–383 (1996)

[3] Bauer, C.: A note on sums of five almost equal prime squares. *Archiv der Mathematik*, **69**, 20–30 (1997)

[4] Liu, M. C., Tsang, K. M.: Small prime solutions of linear equations. In: Number Theory (Eds. J. M. De Konick, C. Levesque), 595–624, Berlin, W. de Gruyter, 1989

[5] Liu, M. C., Tsang, K. M.: Small prime solutions of some additive equations. *Monatshefte Mathematik*, **111**, 147–169 (1991).

[6] Liu, J. Y., Zhan, T.: Sums of five almost equal prime squares. *Science in China, Series A*, **41**(7), 710–722 (1998)

[7] Harman, G.: Trigonometric sums over primes. *Mathematika*, **28**, 249–254 (1981). *Math.*, **34**, 1365–1377 (1982)

[8] Pan, C. D., Pan, C.B.: Goldbach conjecture, Beijing, Science Press, 1992

[9] Titchmarsh, E. C.: The Theory of the Riemann Zeta-Function, Second edition, Oxford, Clarendon Press, 1986

[10] Pan, C. D., Pan, C. B.: Analytic number theory (Chinese) Beijing, Science Press, 1992

[11] Leung, M. C., Liu, M. C.: On generalized quadratic equations in three prime variables. *Monatshefte Mathematik*, **115**, 133–169 (1993)

[12] Gallagher, P. X.: A large sieve density estimate near $\sigma = 1$. *Inventiones Math.*, **11**, 329–339 (1970)

[13] Zhan, T.: On the representation of a large odd integer as a sum of three almost equal primes. *Acta Mathematica Sinica, New Series,* **7**(3), 259–272 (1991)

[14] Bauer, C., Liu, M. C., Zhan, T.: On sums of three primes, *Journal of Number Theory*, **85**(2), 336–359 (2000).