ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 4, Winter 2001

AN IMPROVEMENT ON A THEOREM OF THE GOLDBACH-WARING TYPE

CLAUS BAUER

ABSTRACT. Let p_i, $2 \le i \le 5$ be prime numbers. It is proved that all but $\ll x^{19193/19200+\varepsilon}$ positive even integers N smaller than x can be represented as

$$N = p_1^2 + p_2^3 + p_3^4 + p_4^5.$$

1. Introduction and statement of results. I.M. Vinogradov [**14**] proved the ternary Goldbach-conjecture in 1937. Its method was successfully applied to different problems in additive prime number theory by various mathematicians. Among them Prachar established in 1952, [**11**] the following result: *There exists a constant $c > 0$ such that all but $\ll x(\log x)^{-c}$ even integers N smaller than x are representable as*

$$N = p_1^2 + p_2^3 + p_3^4 + p_4^5 \tag{1.1}$$

for prime numbers p_i.

The author could improve upon this result in [**1**] by giving the following estimate: *There exists a positive number δ such that all but*

$$\ll x^{1-\delta}$$

positive even integers $N \le x$ are representable as in (1.1).

Here the constant δ is very small and its value depends on the existence of the possible Siegel-zero (see [**3**]) of the Dirichlet series $L(s, \chi)$. Using a method first developed in [**2**] we will improve on this estimate by showing the following theorem:

Theorem. *All but $\ll x^{19193/19200+\varepsilon}$ positive even integers smaller than x can be represented as in* (1.1).

AMS *Mathematics Subject Classification.* 11P32, 11L07.
Received by the editors on August 7, 2000.

Using the circle method the main difficulties arise on the major arcs, where we apply mean value estimates for Dirichlet polynomials and power moments of L-functions. Compared to [**1**] no special attention is paid to the possible Siegel zero and the Deuring-Heilbronn phenomena is not used.

2. Notation and structure of the proof. We will choose our notation similar to the one in [**8**]. By k we will always denote an integer $k \in \{2,3,4,5\}$, by p we denote a prime number and L denotes $\log x$. c is an effective positive constant and ε will denote an arbitrarily small positive number; both of them may take different values at different occasions. For example, we may write

$$L^c L^c \ll L^c, \qquad x^\varepsilon L^c \ll x^\varepsilon.$$

$d_2(n)$ denotes the number of divisors of n and $[a_1, \dots, a_n]$ denotes the least common multiple of the integers $a_1, \dots, a_n$. Be further

$$r \sim R \Longleftrightarrow R/2 < r \le R, \quad \sum_{\chi \bmod q}^{*} = \sum_{\substack{x \bmod q \\ x \text{primitive}}}, \quad \sum_{1 \le a \le q}^{*} = \sum_{\substack{1 \le a \le q \\ (a,q)=1}}^{q}.$$

$$P = N^{(7/150-\varepsilon)}, \quad Q = NP^{-1}L^{-E}, \quad (E > 0 \quad \text{will be defined later}),$$

and

$$\mu = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - 1.$$

We define for any characters $\chi, \chi_j \pmod q$, $q \le P$, and a fixed integer N:

$$C_k(a,\chi) = \sum_{l=1}^{q} \chi(l) e\left(\frac{al^k}{q}\right), \qquad C_k(a,\chi_0) = C_k(a,q)$$

$$Z(q,\chi_2,\chi_3,\chi_4,\chi_5) = \sum_{h=1}^{q}{}^{*} e\left(\frac{-hN}{q}\right) \prod_{k=2}^{5} C_k(h,\chi_k),$$

$$Y(q) = Z(q,\chi_0,\chi_0,\chi_0,\chi_0), \qquad A(q) = \frac{Y(q)}{\phi^4(q)},$$

$$S_k(\lambda,\chi) = \sum_{\sqrt[k]{x}/2^{k+1} \le n \le \sqrt[k]{x}} \Lambda(n)\chi(n)e(n^k\lambda),$$

$$T_k(\lambda) = \sum_{\sqrt[k]{x}/2^{k+1} \leq n \leq \sqrt[k]{x}} e(n^k \lambda),$$
$$W_k(\lambda, \chi) = S_k(\lambda, \chi) - E_0 T_k(\lambda),$$
$$E_0 = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases}$$

Using the circle method we define the major arcs M and minor arcs m as follows:

$$M = \sum_{q \leq P} \sum_{a=1}^{q}{}^{*} I(a, q), I(a, q) = \left[\frac{a}{q} - \frac{1}{Qq}, \frac{a}{q} + \frac{1}{Qq}\right],$$
$$m = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus M.$$

Let

$$R(N) = \sum_{\substack{\sqrt[k]{x}/2^{k+1} \leq n_k \leq \sqrt[k]{x} \\ k \in \{2, \ldots, 5\} \\ n_2^2 + \cdots + n_5^5 = N}} \Lambda(n_2) \cdots \Lambda(n_5).$$

Then we find

$$\begin{aligned} R(N) &= \int_{1/Q}^{1+1/Q} e(-N\alpha) \prod_{k=2}^{5} S_k(\alpha)\, d\alpha \\ &= \left(\int_M + \int_m\right) e(-N\alpha) \prod_{k=2}^{5} S_k(\alpha)\, d\alpha \\ &=: R_1(N) + R_2(N). \end{aligned} \tag{2.1}$$

Using Theorem 1 in [**5**] and Lemma 3 in [**11**], we obtain

$$\begin{aligned} \sum_{x/2 \leq N < x} |I_2(N)|^2 &\leq \max_{\alpha \in m} |S_5(\alpha)|^2 \int_m |S_2(\alpha) S_3(\alpha) S_4(\alpha)|^2 \\ &\ll x^{2\mu+1+\varepsilon} P^{-1/128}, \end{aligned}$$

from which we derive that

$$I_2(N) \ll N^{\mu} L^{-1000} \tag{2.2}$$

for all but $\ll x^{1+2\varepsilon} P^{-1/128} < x^{19193/19200+3\varepsilon}$ even integers $x/2 \leq N < x$. In Sections 3–5 we will show that, for any given $G > 0$,

$$R_1(N) = \frac{1}{120} P_0 \sum_{q \leq P} A(q) + O(x^\mu L^{-G}), \tag{2.3}$$

where

$$x^\mu \ll P_0 := \sum_{\substack{m_1+m_2+m_3+m_4=N \\ x/2^{k+1}<m_k \leq x}} \frac{1}{m^{1-(1/k)}} \ll x^\mu \quad \text{for } N \in (x/2, x]. \tag{2.4}$$

In Section 6 we will derive from (2.3) that for all but $\ll x^{443/450+\varepsilon}$ positive even integers $x/2 < N \leq x$, the following holds

$$R_1(N) = \frac{1}{120} P_0 \prod_{p \leq P} s(p) + O(x^\mu L^{-G}). \tag{2.5}$$

Using that

$$\prod_{p \leq P} s(p) \gg (\log P)^{-960}$$

(see [**1**, Lemma 4.5]) the theorem follows from (2.1), (2.2), (2.4) and (2.5).

3. The major arcs.

We will make use of the following lemmas:

Lemma 3.1. *If $(a, q) = 1$, then*

$$C_k(a, \chi_q) \ll q^{1/2+\varepsilon}.$$

Proof. This is contained in Lemmas 5.1 and 5.2 in [**9**].

Lemma 3.2. *Let $f(x), g(x)$ and $f'(x)$ be three real differentiable and monotonic functions in the interval $[a, b]$ and $|g(x)| \ll M$.*

(i) *If $|f'(x)| \gg m > 0$, then*

$$\int_a^b g(x)e(f(x))\, dx \ll M/m.$$

(ii) *If* $|f''(x)| \gg r > 0$, *then*

$$\int_a^b g(x)e(f(x))\,dx \ll M/r^{1/2}.$$

(iii) *If* $|f'(x)| \le \theta < 1$, $g(x)$, $g'(x) \ll 1$, *then*

$$\sum_{a<n\le b} g(n)e(f(n)) = \int_a^b g(x)e(f(x))\,dx + O\left(\frac{1}{1-\theta}\right).$$

Proof. See Lemma 4.8 in [**13**].

Lemma 3.3. *For primitive characters* $\chi_1 \bmod r_i$, $i = 1, 2, 3, 4$, *and the principal character* $\chi_0 \bmod q$, *we have*

$$\sum_{\substack{q\le P\\ r|q}} \frac{|Z(q, \chi_0\chi_1, \chi_0\chi_2, \chi_0\chi_3, \chi_0\chi_4)|}{\phi^4(q)} \ll r^{-1+\varepsilon}(\log P)^c,$$

where $r = [r_1, r_2, r_3, r_4]$.

Proof. Let J denote the lefthand side in Lemma 3.3, and write $Z(q) = Z(q, \chi_0\chi_1, \chi_0\chi_2, \chi_0\chi_4, \chi_0\chi_4)$. Using Lemmas 4.1 and 4.3 a) in [**1**], we argue as in the proof of Lemma 6.7 in [**7**] and obtain

$$J \ll \sum_{u|a} \frac{|Z(ur)|}{\phi^4(ur)} \sum_{\substack{q\le P/ur\\ (q,r)=1}} |A(q)|,$$

where $a \ll 1$. From Lemma 3.1, we derive

$$\sum_{u|a} \frac{|Z(ur)|}{\phi^4(ur)} \ll r^{-1+\varepsilon}.$$

Lemma 3.3 follows therefore from

Lemma 3.4.

$$\sum_{q\le P}|A(q)|\ll(\log P)^c.$$

Proof. Using Lemmas 4.1, 4.4a) and (4.6) in [**1**], we find

$$\sum_{q\le P}|A(q)|\ll\prod_{p\le P}\left(1+\frac{c}{p}\right)\ll(\log P)^c.$$

Splitting the summation over n in residue classes modulo q we obtain

$$S_k\left(\frac{a}{q}+\lambda\right)=\frac{C_k(a,q)}{\phi(q)}\,T_k(\lambda)+\frac{1}{\phi(q)}\sum_{\chi \bmod q}C_k(a,\chi)W_k(\lambda,\chi)+O(L^2).$$

Thus we obtain from (2.1),

$$R_1(N)=R_1^m(N)+R_1^e(N)+O(x^\mu L^{-G})\quad\text{for any } G>0, \tag{3.1}$$

where

$$R_1^m(N)=\sum_{q\le P}\frac{1}{\phi^4(q)}$$

$$\cdot\sum_{1\le a\le q}^{*}\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}C_k(a,q)e\left(-\frac{a}{q}\,N\right)T_k(\lambda)e(-\lambda N)\,d\lambda,$$

$$R_1^e(N)=\sum_{\substack{k,l=2\\k<l}}^{5}\sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{1\le a\le q}^{*}\int_{-1/Qq}^{1/Qq}\prod_{m\in\{k,l\}}C_m(a,q)T_m(\lambda)$$

$$\cdot\prod_{\substack{o=2\\o\ne k\\o\ne l}}^{5}\sum_{\chi \bmod q}C_0(a,\chi)W_0(\lambda,\chi)e\left(-\frac{a}{q}\,N-\lambda N\right)d\lambda$$

$$+\sum_{k=2}^{5}\sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{1\le a\le q}^{*}\int_{-1/Qq}^{1/Qq}C_k(a,q)T_k(\lambda)$$

$$\cdot\prod_{\substack{l=2\\l\ne k}}^{5}\sum_{\chi \bmod q}C_l(a,q)W_l(\lambda,\chi)e\left(-\frac{a}{q}\,N-\lambda N\right)d\lambda$$

$$+\sum_{q\le P}\frac{1}{\phi^4(q)}\sideset{}{^*}\sum_{1\le a\le q}\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{t}\sum_{\chi \bmod q} \cdot C_k(a,\chi)W_k(\chi,\lambda)e\Big(-\frac{a}{q}N-\lambda N\Big)\,d\lambda,$$

$$=: S_1+S_2+S_3+S_4.$$

We first calculate $R_1^m(N)$. Applying Lemma 3.2 yields

$$\begin{aligned}T_k(\lambda)&=\int_{\sqrt[k]{x}/2^{k+1}}^{\sqrt[k]{x}} e(\lambda u^k)\,du+O(1)\\&=\frac{1}{k}\int_{x/2^{k+1}}^{x} v^{1/k-1}e(\lambda v)\,dv+O(1)\\&=\frac{1}{k}\sum_{x/2k+1<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}+O(1).\end{aligned}$$

Substituting this in $R_1^m(N)$, we see

$$\begin{aligned}R_1^m(N)=\frac{1}{120}\sum_{q\le P}A(q)\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}\Bigg(\sum_{x/2^{k+1}<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}\Bigg)e(-N\lambda)\,d\lambda\\+O\Bigg(\Bigg|\max_{2\le l\le 5}\sum_{q\le P}A(q)\int_{1/Qq}^{-1/Qq}\prod_{\substack{k=2\\k\ne l}}^{5}\sum_{2/2^{k+1}<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}\,d\lambda\Bigg|\Bigg).\end{aligned}$$

Using Lemma 3.3 and the trivial bound

$$\sum_{x/2^{k+1}<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}\ll \min\Bigg(\sqrt[k]{x},\frac{1}{x^{1-(1/k)}|\lambda|}\Bigg) \tag{3.2}$$

we derive

(3.3)

$$\begin{aligned} R_1^m(N) &= \frac{1}{120}\sum_{q\le P} A(q)\int_{-1/2}^{1/2}\prod_{k=2}^{5}\Bigg(\sum_{x/2^{k+1}<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}\Bigg)e(-N\lambda)\,d\lambda \\ &\quad + O\Bigg(\sum_{q\le P}|A(q)|\int_{1/Qq}^{1/2}\frac{1}{x^{3-\mu}|\lambda|^4}\,d\lambda\Bigg) + O(x^{\mu}L^{-G}) \\ &= \frac{1}{120}P_0\sum_{q\le P}A(q) + O((PQ)^3x^{\mu-3}L^c) + O(x^{\mu}L^{-G}) \\ &= \frac{1}{120}P_0\sum_{q\le P}A(q) + O(x^{\mu}L^{-G}), \end{aligned}$$

where P_0 is defined as in (2.4) and E is chosen sufficiently large in $Q = NP^{-1}L^{-E}$. In the sequel $E = E(G)$ is fixed. Now we estimate the terms S_i, $i = 1, 2, 3, 4$. Using Lemma 3.3 we can estimate S_4 in the following way:

$$\begin{aligned} |S_4| &\le \sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{\chi_2 \bmod q}\ \sum_{\chi_3 \bmod q}\ \sum_{\chi_4 \bmod q}\ \sum_{\chi_5 \bmod q} \\ &\quad \cdot |Z(q,\chi_2,\chi_3,\chi_4,\chi_5)|\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}|W_k(\lambda,\chi_j)|\,d\lambda \\ &\le \sum_{r_2\le P}\sum_{r_3\le P}\sum_{r_4\le P}\sum_{\substack{r_5\le P\\ [r_2,r_3,r_4,r_5]\le P}}\ \sideset{}{^*}\sum_{\chi_2 \bmod r_3}\ \sideset{}{^*}\sum_{\chi_3 \bmod r_3}\ \sideset{}{^*}\sum_{\chi_4 \bmod r_4}\ \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \\ &\quad \cdot \int_{-1/Q[r_2,r_3,r_4,r_5]}^{1/Q[r_2,r_3,r_4,r_5]}\prod_{k=2}^{5}|W_k(\lambda,\chi_k)|\,d\lambda \\ &\quad \cdot \sum_{\substack{q\le P\\ [r_2,r_3,r_4,r_5]|q}}\frac{|Z(q,\chi_2\chi_0,\chi_3\chi_0,\chi_4\chi_0,\chi_5\chi_0)|}{\phi^4(q)}, \\ &\ll L^c\sum_{r_2\le P}\sum_{r_3\le P}\sum_{r_4\le P}\sum_{r_5\le P}[r_2,r_3,r_4,r_5]^{-1+\varepsilon} \\ &\quad \cdot \sideset{}{^*}\sum_{\chi_2 \bmod r_2}\ \sideset{}{^*}\sum_{\chi_3 \bmod r_3}\ \sideset{}{^*}\sum_{\chi_4 \bmod r_4}\ \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \end{aligned}$$

$$\cdot \int_{-1/Q[r_2,r_3,r_4,r_5]} \prod_{k=2}^{5} |W_k(\lambda,\chi_k)| \, d\lambda.$$

Using $[r_2,r_3,r_4,r_5] \geq (r_2r_3)^{1/7}(r_4r_5)^{5/14}$, we obtain

$$\begin{aligned} S_4 \ll L^c & \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} \sum_{r_k\leq P} r_k^{-5/14+\varepsilon} \sum_{\chi_k \bmod r_k}{}^{*} \\ & \cdot |W_k(\lambda,\chi_k\,| \max_{|\lambda|\leq 1/Q} \sum_{r_l\leq P} r_l^{-5/14+\varepsilon} \\ & \cdot \sum_{\chi_l \bmod r_l}{}^{*} |W_l(\lambda,\chi_l\,| \sum_{r_m\leq P} r_m^{-1/7+\varepsilon} \sum_{\chi_m \bmod r_m}{}^{*} \\ & \cdot \left(\int_{-1/Qr_m}^{1/Qr_m} |W_m(\lambda,\chi_m|^2 \, d\lambda \right)^{1/2} \\ & \cdot \sum_{r_n\leq P} r_n^{-1/7+\varepsilon} \sum_{\chi_n \bmod r_n}{}^{*} \left(\int_{-1/Qr_n}^{1/Qr_n} |W_n(\lambda,\chi_n|^2 \, d\lambda \right)^{1/2} \\ \ll L^c & \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} I_k(\lambda) \max_{|\lambda|\leq 1/Q} I_l(\lambda) W_m W_n, \end{aligned} \tag{3.4}$$

where

$$I_k(\lambda) = \sum_{r\leq P} r^{-5/14+\varepsilon} \sum_{\chi}{}^{*} |W_k(\lambda,\chi|,$$

$$W_k = \sum_{r\leq P} r^{-1/7+\varepsilon} \sum_{\chi}{}^{*} \left(\int_{-1/Qr}^{1/Qr} |W_k(\lambda,\chi|^2 \, d\lambda \right)^{1/2}.$$

Arguing similarly we obtain

$$\begin{aligned} S_1+S_2+S_3 \ll L^c & \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} |T_k(\lambda)| \\ & \cdot \max_{|\lambda|\leq 1/Q} |T_l(\lambda)| \left(\int_{-1/Q}^{1/Q} |T_m(\lambda)|^2 \, d\lambda \right)^{1/2} W_n \\ & + L^c \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} |T_k(\lambda)| \max_{|\lambda|\leq 1/Q} |T_l(\lambda)| W_m W_n \\ & + L^c \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} |T_k(\lambda)| \max_{|\lambda|\leq 1/Q} I_l(\lambda) W_m W_n. \end{aligned} \tag{3.5}$$

We have trivially

$$\max_{|\lambda|\leq 1/Q} |T_k(\lambda)| \ll x^{1/k}.$$

Using (3.2) we obtain

$$\left(\int_{-1/Q}^{1/Q} |T(\lambda)|^2\, d\lambda\right)^{1/2} \ll x^{(1/k)-(1/2)}.$$

Thus we see from (3.1) and (3.3)–(3.5) that the proof of (2.3) reduces to the proof of the following two lemmas:

Lemma 3.5. *If $P \leq x^{(7/150)-\varepsilon}$ and $2 \leq k \leq 5$,*

$$W_k \ll_B x^{1/k-1/2} L^{-B}$$

for any $B > 0$.

Lemma 3.6. *If $P \leq x^{(7/150)-\varepsilon}$ and $2 \leq k \leq 5$,*

$$\max_{|\lambda|\leq 1/Q} I(\lambda) \ll x^{1/k} L^A$$

for a certain $A > 0$.

For the proof of these lemmas we will appeal to the following lemmas:

Lemma 3.7. *For any $P \geq 1$, $T \geq 1$ and $k = 0, 1$,*

$$\sum_{q\leq P} \sideset{}{^*}\sum_{\chi} \int_{-T}^{T} \left| L^{(k)}\left(\frac{1}{2} + it, \chi\right)\right|^4 dt \ll P^2 T (\log PT)^{4(k+1)}.$$

Lemma 3.8. *For any $P \geq 1$, $T \geq 1$ and any complex numbers a_n*

$$\sum_{q\leq P} \sideset{}{^*}\sum_{\chi} \int_{-T}^{T} \left| \sum_{n=M+N}^{M} a_n \chi(n) n^{-it}\right|^2 dt \ll \sum_{n=M+N}^{M} (P^2 T + n)|a_n|^2.$$

Lemma 3.9. *Let $N^*(\alpha, T, q)$ denote the number of zeros $\sigma + it$ of all L-functions to primitive characters modulo q within the region $\sigma \geq \alpha$, $|t| \leq T$. Then*

$$\sum_{q \leq Q} N^*(\alpha, T, q) \ll T^{12(1-\alpha)/5} (\log QT)^c.$$

The lemmas 3.7–3.9 may be found in [**10**, Chapters 2, 3 and 5].

4. Proof of Lemma 3.5. In order to prove the lemma, it is enough to show that

$$W_{k,R} \ll x^{(1/k)-(1/2)} R^{1/7-\varepsilon} L^{-B}, \tag{4.1}$$

where

$$W_{k,R} = \sum_{r \sim R} \sum_{\chi}{}^{*} \left(\int_{-1/Qr}^{1/Qr} |W_k(\lambda, \chi|^2 \, d\lambda \right)^{1/2}$$

for $R \leq P/2$. Applying Lemma 1 [**4**], we see

(4.2)

$$\begin{aligned}&\int_{-1/Qr}^{1/Qr} |W_k(\lambda, \chi)|^2 \, d\lambda \\ &\ll (QR)^{-2} \int_{x/2^{k+2}}^{x} \Bigg| \sum_{\substack{t < m^k \leq t + Qr \\ x/2^{k+1} < m^k \leq x}} \Lambda(m)\chi(m) - E_0 \sum_{\substack{t < m^k \leq t + Qr \\ x/2^{k+1} < m^k \leq x}} 1 \Bigg|^2 dt.\end{aligned}$$

We set $X = \max(x/2^{k+1}, t)$ and $X + Y = \min(x, t + Qr)$. In the sequel we will treat the case $R > L^D$ and $R \leq L^D$ for a sufficiently large constant $D > 0$ separately. In the first case we apply a slight modification of Heath-Brown's identity [**6**],

$$\begin{aligned}-\frac{\zeta^\iota}{\zeta}(s) = &\sum_{j=1}^{K} \binom{K}{j} (-1)^{j-1} \zeta^\iota(s) \zeta^{j-1}(s) M^j(s) \\ &- \frac{\zeta^\iota}{\zeta}(s)(1 - \zeta(s)M(s))^K,\end{aligned}$$

with $K = 5$ and

$$M(s) = \sum_{n \le x^{1/5k}} \mu(n)$$

to the sum

$$\sum_{X < m^k \le X+Y} .$$

Arguing exactly as in part III, [**15**], we find by applying Heath Brown's identity and Perron's summation formula (see [**13**, Lemma 3.12]) that the inner sum of (4.2) – where always $E_0 = 0$ because of $R > L^D$ and the primitivity of the characters – is a linear combination of $O(L^c)$ terms of the form

$$S_k = \frac{1}{2\pi i} \int_{-T}^{T} F_k\left(\frac{1}{2} + iu, \chi\right) \frac{(X+Y)^{(1/2+iu)/k} - X^{(1/2+iu)/k}}{(1/2) + iu}\, du$$
$$+ O(T^{-1} x^{(1/k)+\varepsilon}),$$

where $2 \le T \le x$,

$$F_k(x, \chi) = \prod_{j=1}^{10} f_{k,j}(s, \chi),$$

$$f_{k,j}(s, \chi) = \sum_{n \in I_{k,j}} a_{k,j}(n) \chi(n) n^{-s},$$

$$a_{k,j}(n) = \begin{cases} \log n \text{ or } 1 & j = 1, \\ 1 & 1 < j \le 5, , \\ \mu(n) & 6 \le 10 \end{cases} \tag{4.3}$$

$$I_j = (N_{k,j}, 2N_{k,j}], \quad 1 \le j \le 10,$$

$$\sqrt[k]{x} \ll \prod_{j=1}^{10} N_{k,j} \ll \sqrt[k]{x}, \quad N_{k,j} \le x^{1/5k}, \quad 6 \le j \le 10.$$

Since

$$\frac{(X+Y)^{(1/k)[(1/2)+iu]} - X^{(1/k)[(1/2)+iu]}}{(1/2) + iu}$$
$$\ll \min(QRx^{(1/2k)-1}, x^{1/2k}(|u| + 1)^{-1})$$

by taking $T = x^{2\varepsilon}P^2(1+|\lambda|x)$ and $T_0 = x(QR)^{-1}$, we conclude that S_k is bounded by

$$\ll QRx^{(1/2k)-1}\int_{-T_0}^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|du + x^{1/2k}\int_{T_0\le|u|\le T}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|\frac{du}{|u|} + x^{1/k}P^{-2},$$

Thus we derive from (4.2) that in order to prove (4.1) it is enough to show that

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)g\right|dt \ll x^{1/2k}R^{1/7-\varepsilon}L^{-B}, \tag{4.4}$$

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_{T_1}^{2T_1}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt \ll x^{1/2k-1}QR^{8/7-\varepsilon}T_1L^{-B}, \quad T_0<|T_1|\le T. \tag{4.5}$$

For the proof of (4.4) and (4.5) we will prove two propositions. We will need the estimate

$$\sum_{n\le x} d_2^k(n) \ll_k xL^{c(k)}. \tag{4.6}$$

We now establish

Proposition 1. *If there exist N_{k,j_1} and N_{k,j_2}, $1\le j_1,\ j_2\le 5$, such that $N_{k,j_1}N_{k,j_2}\ge P^{12/7+3\varepsilon}$, then* (4.4) *is true.*

Proof. We suppose without loss of generality that $j_1=1$, $a_1(n)=\log n$ and $j_2=2$, $a_2(n)=1$. Arguing exactly as in the proof of Proposition 1 in [**15**], we find

$$f_{k,1}\left(\frac{1}{2}+it,\chi\right) \ll L\left(\int_{-x^{1/k}}^{x^{1/k}}\left|L'\left(\frac{1}{2}+it+iv,\chi\right)\right|^4\frac{dv}{1+|v|}\right)^{1/4} + L,$$

and so we find by using Lemma 3.7,

$$\begin{aligned}
&\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|f_1\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\\
&\ll L^4\int_{-x^{1/k}}^{x^{1/k}}\frac{dv}{1+|v|}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_v^{T_0+v}\left|L'\left(\frac{1}{2}+it,\chi\right)\right|^4 dt+T_0R^2L^4\\
&\ll L^5\max_{|N|\le x^{1/k}}\int_{N/2}^{N}\frac{dv}{1+|v|}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_v^{T_0+v}\left|L'\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\\
&\quad+T_0R^2L^4\\
&\quad+L^5\max_{|N|\le x^{1/k}}N^{-1}\int_0^{T_0}dt\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{(N/2)+t}^{N+t}\left|L'\left(\frac{1}{2}+iv,\chi\right)\right|^4 dv\\
&\quad+T_0R^2L^4\\
&\ll R^2T_0L^c.
\end{aligned}$$

Using Lemma 3.8, (4.6) and Holder's inequality, we obtain

$$\begin{aligned}
&\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt\\
&\ll\left(\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|f_{k,1}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/4}\\
&\quad\cdot\left(\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|f_{k,2}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/4}\\
&\quad\cdot\left(\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|\prod_{j=3}^{10}f_{k,j}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/2}\\
&\ll(R^2T_0)^{1/2}\left(R^2T_0+\frac{x^{1/k}}{N_{k,1}N_{k,2}}\right)^{1/2}L^c\\
&\ll x^{1/2k}R^{1/7-\varepsilon}L^{-B},
\end{aligned}$$

by the definition of T_0 and the condition of the proposition.

Proposition 2. *Let $J = \{1, \dots, 10\}$. If J can be divided into two nonoverlapping subsets J_1 and J_2 such that*

$$\max\left(\prod_{j\in J_1} N_{k,j}, \prod_{j\in J_2} N_{k,j}\right) \ll x^{1/k} P^{-(12/7)-3\varepsilon}$$

then (4.4) *is true.*

Proof. let

$$\begin{aligned} F_{k,i}(s,\chi) &= \prod_{j\in J_i} f_{k,j}(s,\chi) \\ &= \sum_{n\ll M_i} b_i(n)\chi(n)n^{-s}, \\ b_i(n) &\ll d_2^c(n), \quad i = 1, 2, \end{aligned}$$

where $M_i = \prod_{j\in J_i N_{k,j}}$, $i = 1, 2$. Applying Lemma 3.8, (4.3) and (4.6) we see

$$\begin{aligned} &\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt \\ &\quad\ll \left(\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F_{k,1}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/2} \\ &\quad\quad\cdot\left(\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F_{k,2}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/2} \\ &\quad\ll (R^2T_0+M_1)^{1/2}(R^2T_0+M_2)^{1/2} \\ &\quad\ll R^2T_0 + x^{1/2k}RP^{-(6/7)-(3/2)\varepsilon}T_0^{1/2} + x^{1/2k}L^c. \end{aligned}$$

This proves the proposition because of $R > L^D$. Whereas for the proof of the proposition an estimate $P \ll x^{(7/130)-\varepsilon}$ would have been enough, we need the estimate $P \le x^{(7/150)-\varepsilon}$ in the following. Now we can prove (4.4). In view of Proposition 1 we assume

$$N_{k,i}N_{k,j} \le P^{12/7+3\varepsilon} \le x^{2/5k}, \quad 1 \le i, \quad j \le 5, \quad i \ne j.$$

Therefore, we see from (4.3) that there exists at most one $N_{k,j}$, $1 \leq j \leq 10$, with $N_{k,j} \geq x^{1/5k}$. Suppose such a $N_{k,j}$ is N_{k,j_0} if it exists (otherwise $N_{k,j_0} = 1$). Reorder the other $N_{k,j}$ as follows:

$$N_{k,j_1} \geq N_{k,j_2} \geq \cdots \geq N_{k,j_K}, \quad K = 9 \text{ or } 10.$$

We find an integer $1 \leq l \leq K-1$ such that

$$N_{k,j_0} N_{k,j_1} \ldots N_{k,j_{l-1}} \leq x^{2/5k} \quad \text{and} \quad N_{k,j_0} N_{k,j_1} \ldots N_{k,j_l} \geq x^{2/5k}.$$

Taking $M_1 = N_{k,j_0} N_{k,j_1} \ldots N_{k,j_l}$ and $M_2 = N_{k,j_{l+1}} \ldots N_{k,j_K}$, we have

$$M_1 \ll x^{2/5k} N_{k,j_l} \leq x^{3/5k} \quad \text{and} \quad M_2 \ll x^{1/5k} M_1^{-1} \ll x^{3/5k}.$$

The sets M_1 and M_2 satisfy the conditions of Proposition 2 and therefore (4.4) is proved. The proof of (4.5) goes along the same lines. (4.1) is now proved in the case $R > L^D$. If $R \leq L^D$ we can estimate the sum on the righthand side of (4.2) by using the zero expansion of the von Mangoldt function:

$$\begin{aligned}
&\sum_{\substack{t<m^k\leq t+Qr\\ x/2^k<m^k\leq x}} \Lambda(m)\chi(m) - E_0 \sum_{\substack{t<m^k\leq t+Qr\\ x/2^k<m^k\leq x}} 1 \\
&\qquad = \sum_{X<m^k\leq X+Y} \Lambda(m)\chi(m) - E_0 \sum_{X<m^k\leq X+Y} 1 \\
&\qquad \ll \sum_{|\mathrm{Im}\,\rho|\leq x^{1/3k}} \left| \frac{(X+Y)^{\rho/k}}{\rho} - \frac{X^{\rho/k}}{\rho} \right| + O(x^{2/3k}L^2) \\
&\qquad \ll QRx^{(1/k)-1} \sum_{|\mathrm{Im}\,\rho|\leq x^{1/3k}} x^{\beta-1/k} + O(x^{2/3k}L^2),
\end{aligned}$$

where ρ runs over the nontrivial zeros of the L-function corresponding to $\chi \bmod r$ with $|\mathrm{Im}\,\rho| \leq x^{1/3k}$ and $\beta = \mathrm{Re}\,\rho$. Applying Lemma 3.9 and the fact that $L(\sigma+it,\chi)$ with $\chi \bmod r \leq L^D$ has no zeros in the region (see [**12**], VIII Satz 6.2)

$$\sigma \geq 1 - \delta(T) := 1 - \frac{c_0}{\log r + (\log(T+2))^{4/5}}, \quad |t| \leq T,$$

where c_0 is an absolute constant and taking $T = x^{1/3k}$ we obtain from (4.2)

$$\begin{aligned}\int_{-1/Qr}^{1/Qr} &|W_k(\lambda,\chi)|^2\,d\lambda \\ &\ll x^{(2/k)-1}\Big(\sum_{|\mathrm{Im}\,\rho|\le x^{1/3k}} x^{(\beta-1)/k}\Big)^2 + (QR)^{-2}x^{1+(4/3k)}L^4 \\ &\ll x^{(2/k)-1}L^c\Big(\max_{(1/2)\le\beta\le 1-\delta(T)} x^{(4/5k)(1-\beta)}x^{(1/k)(\beta-1)}\Big)^2 \\ &\quad + P^2x^{(4/3k)-1}L^{2E+4} \\ &\ll x^{(2/k)-1}\exp(-cL^{1/5}).\end{aligned}$$

This gives (4.1) for $R \le L^D$.

5. Proof of Lemma 3.6. To prove the lemma it is enough to show that

$$\max_{R\le P/2}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}|W_k(\lambda,\chi_r)| \ll x^{1/k}R^{(5/14)-\varepsilon}L^A,$$

uniformly for $|\lambda| \le Q^{-1}$. Arguing as in the section before – we do not have to apply Gallagher's lemma here – we find

$$\begin{aligned}W_k(\lambda,\chi) \ll L^c \max_{I_{a_1},\dots,I_{a_{2k+1}}}\Big|&\int_{-T}^{T} F\Big(\frac{1}{2}+it,\chi\Big)\,dt \\ &\cdot\int_{x/2^{k+1}}^{x} u^{(1/2k)-1}e\Big(\frac{t}{2k\pi}\log u+\lambda u\Big)\,du\Big| + x^{1/k}P^{-1},\end{aligned}$$

for $T = P^3$. Estimating the inner integral by Lemma 3.2, we obtain

$$\begin{aligned}\int_{x/2^{k+1}}^{x} &u^{(1/2k)-1}e\Big(\frac{t}{2k\pi}\log u+\lambda u\Big)\,du \\ &\ll x^{(1/2k)-1}\min\Big(\frac{x}{\sqrt{|t|+1}},\frac{x}{\min_{x/2^{k+1}<u\le x}}\,|t+2k\pi\lambda u|\Big).\end{aligned}$$

Taking $T_0 = 4k\pi xQ^{-1}$, we conclude that in order to prove this lemma it is enough to prove that for $P \le x^{(7/150)-\varepsilon}$ and $2 \le k \le 5$, the following holds

(5.1)
$$\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt \ll x^{1/2k}T_0^{1/2}R^{5/14-\varepsilon}L^c,$$

(5.2)
$$\sum_{r\sim R}\sum_{\chi}{}^{*}\int_{T_1}^{2T_1}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt \ll x^{1/2k}R^{5/14-\varepsilon}T_1L^c,$$
$$T_0<|T_1|\le T.$$

These estimates are shown in the same way as (4.4) and (4.5). here the condition $P\le x^{(7/150)-\varepsilon}$ is needed. Two propositions analogous to Propositions 1 and 2 are proved:

Proposition 3. *If there exist* N_{k,j_1} *and* N_{k,j_2}, $1\le j_1,\ j_2\le 5$, *such that* $N_{k,j_1},N_{k,j_2}\ge P^{9/7+3\varepsilon}$, *then* (5.1) *is true.*

Proposition 4. *Let* $J=\{1,\dots,10\}$. *If* J *can be divided into two nonoverlapping subsets* J_1 *and* J_2 *such that*

$$\max\left(\prod_{j\in J_1}N_{k,j},\prod_{j\in J_2}N_{k,j}\right)\ll x^{1/k}P^{-(9/7)-3\varepsilon},$$

then (5.1) *is true.*

Remark. Here we do not need to treat the case $R>L^D$ separately because we do not have to save a factor L^{-B}.

6. The singular series. We now derive (2.5) from (2.3). In the sequel we write $A(q,N)$ instead of $A(q)$ and $s(p,N)$ instead of $s(p)$ because we will argue for variable N.

Lemma 6.1. *For* $P\le x^{(7/150)-\varepsilon}$, *we have*

$$\sum_{N\le x}\left|\prod_{p\le P}s(p,N)-\sum_{q\le P}A(q,N)\right|\ll xP^{-(1/3)+\varepsilon}, \tag{6.1}$$

which implies that for all but $\ll x^{1+2\varepsilon}P^{-1/3}$ *even integers* N *with* $1 \le N \le x$, *the following holds*

$$\prod_{p\le P} s(p,N) = \sum_{q\le P} A(q,N) + O(x^{-\varepsilon}). \tag{6.2}$$

From here, (2.5) follows.

Proof. Equation (6.1) was proved in Lemma 5.1 in [**1**] for a sufficiently small ε for P as large as x^{ε}. We show that it also holds for $x^{\varepsilon} < P \le x^{(7/150)-\varepsilon}$. We argue exactly as in the proof of Lemma 5.1 in [**1**], but here we set: $V := \exp(\log x \log P/\log\log x)$ and $v = 3\log\log x/4\log P$. Denoting the lefthand side in (6.1) by J, we follow the proof of Lemma 5.1 in [**1**]:

$$\begin{aligned} J &\ll xV^{-v}L^{cL^{1/2}} + x^{1+\varepsilon}P^{-1/3} + x^{7/8+\varepsilon} + x^{(31/40)+\varepsilon} \\ &\quad\cdot \sum_{10\le m\le (2+\varepsilon)\log P/\log\log x} (m\log(xe))^m \\ &\ll x^{7/8+\varepsilon} + x^{1+\varepsilon}P^{-1/3} + x^{(31/40)+\varepsilon} \\ &\quad\cdot \sum_{10\le m\le (2+\varepsilon)\log P/\log\log x} (m(\log(xe))^m. \end{aligned} \tag{6.3}$$

For the calculation of the last sum, we have used $x^{(m-1)/2} \le V$ and therefore $m \le (2+\varepsilon)\log P(\log\log x)^{-1}$ for a sufficiently large x. We obtain as an upper bound:

$$\begin{aligned} &\ll P^{2+\varepsilon} \sum_{10\le m\le (2+\varepsilon)\log P/\log\log x} (\log(xe))^m \\ &\ll P^{2+\varepsilon}\exp((2+\varepsilon)\log P\log\log(xe)/\log\log x)\log P/\log\log x \\ &\ll P^{2+\varepsilon}\exp((2+2\varepsilon)\log P)\log P \ll P^{4+3\varepsilon}. \end{aligned} \tag{6.4}$$

We derive from (6.3) and (6.4)

$$\begin{aligned} J &\ll x^{7/8+\varepsilon} + x^{1+\varepsilon}P^{-1/3} + x^{(31/40)+\varepsilon}P^{4+3\varepsilon}L^c \\ &\ll x^{1+\varepsilon}P^{-1/3}. \end{aligned}$$

This completes the proof of Lemma 6.1.

REFERENCES

1. C. Bauer, *On a problem of the Goldbach-Waring type*, Acta Math. Sinica New Ser. **14** (1998), 223–234.

2. C. Bauer, M.C. Liu and T. Zhan, *On sums of three primes*, J. Number Theory, to appear.

3. H. Davenport, *Multiplicative number theory*, 2nd ed., Springer-Verlag, Chicago, 1980.

4. P.X. Gallagher, *A large sieve density estimate near* $\sigma = 1$, Invent. Math. **11** (1970), 329–339.

5. G. Harman, *Trigonometric sums over primes*, Mathematika **28** (1981), 249–254.

6. D.R. Heath Brown, *Prime numbers in sort intervals and a generalized Vaughan's identity*, Canadian J. Math. **34** (1982), 1365–1377.

7. M.C. Leung and M.C. Liu, *On generalized quadratic equations in three prime variables*, Monatsh. Math. **115** (1993), 133–169.

8. M.C. Liu and K.M. Tsang, *Small prime solutions of linear equations*, in *Number theory* (J.-M. De Konick and C. Levesque, eds.), W. de Gruyter, Berlin, 1989.

9. H.L. Montgomery and R.C. Vaughan, *On the exceptional set in Goldbach's problem*, Acta Arith. **27** (1975), 353–370.

10. Chengdong Pan and Chengbiao Pan, *Goldbach's conjecture*, Science Press, Beijing, 1992.

11. K. Prachar, *Über ein Problem von Waring-Goldbach'schen Typ.*, Monatsh. Math. **57** (1953), 66–74.

12. ———, *Primzahlverteilung*, Springer-Verlag, Berlin, New York, 1978.

13. E.C. Titchmarsh, *The theory of the Riemann zeta function*, 2nd ed., Clarendon Press, Oxford, 1986.

14. I.M. Vinogradov, *Estimation of certain trigonometric sums with prime variables*, Izv. Akad. Nauk SSSR Ser. Mat. **3** (1939), 371–398.

15. T. Zhan, *On the representation of a large odd integer as a sum of three almost equal primes*, Acta Math. Sinica **7** (1991), 259–272.

Tellabs Operations Inc., One Kendall Square, Bldg. 100, Suite 121, Cambridge, MA 02139
E-mail address: claus.bauer@tellabs.com