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ON THE GOLDBACH CONJECTURE IN ARITHMETIC PROGRESSIONS

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ABSTRACT. It is proved that for a given integer N and for all but $\ll (\log N)^B$ prime numbers $k \leq N^{5/48-\varepsilon}$ the following is true: For any positive integers b_i, $i \in \{1,2,3\}$, $(b_i,k)=1$ that satisfy $N \equiv b_1+b_2+b_3 \pmod{k}$, N can be written as $N=p_1+p_2+p_3$, where the p_i, $i \in \{1,2,3\}$ are prime numbers that satisfy $p_i \equiv b_i \pmod{k}$.

1. Introduction. Vinogradov [**17**] has proved that every sufficiently large odd positive integer can be written as the sum of three primes. This theorem has been generalized in many ways. In 1953, Ayoub [**1**] proved the following result: *If k is a fixed positive integer, b_i, $i=1,2,3$, are integers with $(b_i,k)=1$ and $J(N;k,b_1,b_2,b_3)$ is the number of solutions of the equation*

$$\begin{cases} N = p_1+p_2+p_3, \\ p_j \equiv b_j \pmod{k}, \end{cases}$$

then

$$J(N;k,b_1,b_2,b_3) = (N;k)\,\frac{N^2}{2\log^3 N}\,(1+o(1)),$$

where for odd integer $N \equiv b_1+b_2+b_3 \pmod{k}$,

$$\sigma(N,k) = \frac{C(k)}{k^2}\prod_{p|k}\frac{p^3}{(p-1)^3+1}\prod_{\substack{p|N\\ p\nmid k}}\frac{(p-1)((p-1)^2-1)}{(p-1)^3+1} \times \prod_{p>2}\left(1+\frac{1}{(p-1)^3}\right),$$

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where all $p > 2$, $C(k) = 2$ *for odd* k *and* $C(k) = 8$ *for even* k.

Using Ayoub's method, one can prove this result for all $k \le \log^A N$ for an arbitrary $A > 0$ for all sufficiently large odd integers N. Liu and Zhan [**11**] as well as the first author [**2**] improved upon Ayoub's result by proving the following theorem:

For $N \equiv b_1 + b_2 + b_3 \pmod{k}$ *and an odd* N *sufficiently large, there holds*

$$J(N; k, b_1, b_2, b_3) > 0 \tag{1.1}$$

for all $k \le N^\delta$, *where* δ *is a very small, positive constant.*

In [**10**], it was shown that (1.1) holds for all $k \le R = N^{(1/8)-\varepsilon}$ with at most $\ll R(\log N)^{-A}$ exceptions for any $A > 0$. Liu proved in [**7**] that if k is restricted to be a prime number, R can be chosen as large as $N^{3/20}(\log N)^{-A}$ for any $A > 0$. Here we give a result that improves on the result in [**7**] by obtaining a significantly smaller set of exceptional modules k at the cost of a smaller upper bound R:

Theorem 1. *Let* $R = N^{5/48-\varepsilon}$. *Then the inequality* (1.1) *holds for all prime numbers* $k \le R$ *with at most* $O((\log N)^B)$ *exceptions for a certain* $B > 0$.

The improvement in this paper compared to previous work is due to two innovations. First, we apply a technique previously used in [**9**] to our problem. Second, as a main contribution of our paper, we exactly calculate the contribution of N-*exceptional* zeros that we define in the following. We set

$$L = \log N, \qquad L_2 = \log\log N, \qquad L(s,\chi) = \sum_{n\ge 1} \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character. For a prime number k, $k \le N$, and a fixed positive integer V, we define

$$P_k = \{m \in \mathbf{N} : m \equiv 0 \pmod{k}\}, \quad I_V = \left[k, kL^V\right] \bigcup \left[k^2, k^2L^V\right],$$
$$A_k = P_k \cap I_V.$$

We call a Dirichlet character χ to a module q, $q \le N$, an N-*exceptional character* if there exists at least one complex number $s = \sigma + it$ such that

$$\sigma > 1 - \frac{EL_2}{L}, \quad |t| \le N, \quad L(s,\chi) = 0, \tag{1.2}$$

where E is a fixed, positive number to be defined later. We call s an N-*exceptional zero* and we call an integer q an N-*exceptional integer* if there exists an N-*exceptional character* χ modulo q.

We note that the concept of N-exceptional zeros has earlier been applied to other problems in additive prime number theory in [**18**] and [**3**]. However, the exact definitions of the N-exceptional zeros in both papers differ from the definition given here and, indeed, the sets of N-exceptional zeros defined here and in [**18**] and [**3**] have no common elements.

Theorem 1 is a direct consequence of Theorems 2 and 3.

Theorem 2. *For a given prime number $k \le N^{5/48-\varepsilon}$, if none of the integers $q \in A_k$ is N-exceptional, then* (1.1) *is true for this k.*

Theorem 3. *There are at most $O\left((\log N)^B\right)$ prime numbers k, $1 \le k \le N$, such that at least one of the integers $q \in A_k$ is N-exceptional. Here, B is a fixed positive constant.*

2. Outline of the proof of Theorem 2 and treatment of the minor arcs. In the sequel, $[a_1, \dots, a_n]$ denotes the least common multiple of the integers $a_1, \dots, a_n$. c is an effective positive constant and ε will denote an arbitrarily small positive number; both of them may take different values at different occasions. For example, we may write

$$L^c L^c \ll L^c, \quad N^\varepsilon L^c \ll N^\varepsilon.$$

We use the familiar notations

$$r \sim R \Longleftrightarrow R < r \le 2R,$$
$$\sideset{}{^*}\sum_{\chi \bmod q} := \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}}, \qquad \sideset{}{^*}\sum_{1 \le a \le q} := \sum_{\substack{1 \le a \le q \\ (a,q)=1}}.$$

We know from [**1**] that Theorem 2 holds true for $k \le L^H$ for any $H > 0$. Therefore, we assume throughout the document that

$$k > L^H \tag{2.1}$$

for a fixed $H > 0$ to be determined later. χ_q denotes a character modulo q and $\chi_{q,0}$ is the principal character modulo q. We write $e(\alpha) = e^{2\pi i \alpha}$ and the variables p and k always denote prime numbers. We keep k fixed throughout this paper. If $p^m | q$, but $p^{m+1} \nmid q$, we write $p^m || q$. We define for any three positive integers r_i, $i \in \{1, 2, 3\}$ that satisfy $k^3 \nmid r_i$,

$$s_i = \begin{cases} r_i & \text{if } k \nmid r_i, \\ r_i/k & \text{if } k || r_i, \\ r_i/k^2 & \text{if } k^2 || r_i. \end{cases} \tag{2.2}$$

Setting $[r_1, r_2, r_3] = r$ and $[s_1, s_2, s_3] = s$, this implies for $k^m || r$, $m \le 2$:

$$r = sk^m. \tag{2.3}$$

For a positive integer q and a character χ modulo q, let

$$k_q = (k, q), \quad R(N) = \sum_{\substack{N/4 \le n_i < N \\ n \equiv b_i \pmod{k} \\ n_1 + n_2 + n_3 = N}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3),$$

$$C(\chi, q, h, b, a) = \sum_{\substack{m=1 \\ m \equiv b \pmod{h}}}^{q} \chi(m)\, e\left(\frac{ma}{q}\right),$$

$$Z(N, q, k_q, \chi_1, \chi_2, \chi_3) := \frac{1}{\phi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C(\chi_1, q, k_q, b_1, a)\, C(\chi_2, q, k_q, b_2, a) \times C(\chi_3, q, k_q, b_3, a) e\left(\frac{-aN}{q}\right),$$

$$A(N, q, k_q) = Z\left(N, q, k_q, \chi_{(q/k_q),0}, \chi_{(q/k_q),0}, \chi_{(q/k_q),0}\right),$$

$$T(\lambda) = \sum_{N/4 < n \le N} e(\lambda n).$$

As we always argue for fixed variables N and k, denote by

(2.4)
$$S(\lambda, b_i) = \sum_{\substack{N/4<n\le N\\ n\equiv b_i \pmod k}} \Lambda(n)e(n\lambda), \quad S(\lambda,\chi) = \sum_{N/4<n\le N} \Lambda(n)e(n\lambda)\chi(n),$$
$$W(\lambda,\chi) = S(\lambda,\chi) - E_0(\chi)T(\lambda), \quad E_0(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise,}\end{cases}$$
$$P_1 = k^{4/3}L^{3G}, \qquad P_2 = k^2L^{3G}, \qquad Q = Nk^{-2}L^{-4G},$$

where the constant $G \ge 8$ will be specified later. Using the circle method, we define the major arcs $M = E_1(k) \cup E_2(k)$ as in [**7**]:

$$E_1(k) = \bigcup_{\substack{q\le P_1\\ k\nmid q}} \bigcup_{\substack{a=1\\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right],$$
$$E_2(k) = \bigcup_{\substack{q\le P_2\\ k\mid q}} \bigcup_{\substack{a=1\\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right],$$

We define the minor arcs m as $m = [(1/Q), 1 + (1/Q)] \setminus M$. Writing $\alpha = (a/q) + \lambda$, we use Dirichlet's theorem on rational approximation and find that $m \subset E_3(k) \cup E_4(k)$, where

$$E_3(k) = \left\{\alpha = \frac{a}{q} + \lambda : P_1 < q < Q,\ k\nmid q,\ |\lambda| \le \frac{1}{qQ}\right\},$$
$$E_4(k) = \left\{\alpha = \frac{a}{q} + \lambda : P_2 < q < Q,\ k|q,\ |\lambda| \le \frac{1}{qQ}\right\}.$$

We see

(2.5)
$$\begin{aligned} R(N) &= \int_{1/Q}^{1+(1/Q)} e(-N\alpha) \prod_{i=1}^{3} S(\alpha, b_i)\, d\alpha \\ &= \left(\sum_{i=1}^{2} \int_{E_i(k)}\right) e(-N\alpha) \prod_{i=1}^{3} S(\alpha, b_i)\, d\alpha \\ &\quad + O\left(\sum_{i=3}^{4} \int_{E_i(k)} \left|\prod_{i=1}^{3} S(\alpha, b_i)\right| d\alpha\right) \\ &=: R_1(N) + R_2(N) + O\left(R_3(N) + R_4(N)\right). \end{aligned}$$

To estimate the contribution of the integral over m, we quote the following lemma from [**7**]:

Lemma 2.1. *Let $A > 0$ be arbitrary and $\alpha \in E_3(k) \cup E_4(k)$. If in (2.4) $G = G(A)$ is chosen sufficiently large, then*

$$S(\alpha, b) \ll \frac{N}{kL^{A+1}}.$$

We derive from Lemma 2.1 and Dirichlet's lemma on rational approximation the following estimate:

$$\begin{aligned} &\int_{E_3(k)\cup E_k(4)} |S(\alpha,b_1)S(\alpha,b_2)S(\alpha,b_3)|\ d\alpha \\ &\qquad \ll \max_{\alpha\in E_3(k)\cup E_4(k)} |S(\alpha,b_1)| \left(\int_0^1 |S(\alpha,b_2)|^2\,d\alpha\right)^{1/2} \\ &\qquad\quad \times \left(\int_0^1 |S(\alpha,b_3)|^2\ d\alpha\right)^{1/2} \\ &\qquad \ll \frac{N^2}{k^2L^A}. \end{aligned} \tag{2.6}$$

In the following sections, we shall show that, under the condition of Theorem 2,

$$R_1(N) + R_2(N) = \sigma(N,k)\,\frac{N^2}{32} + O\left(N^2k^{-2}L^{-A}\right), \tag{2.7}$$

for any $A > 0$ and where $\sigma(N,k)$ is defined as in (1.2). Using

$$\frac{k}{\phi^3(k)} \gg \sigma(N,k) \gg \frac{k}{\phi^3(k)},$$

Theorem 2 follows from (2.5), (2.6) and (2.7).

3. Preliminary lemmas.

Lemma 3.1. *Let $f(x)$, $g(x)$ and $f'(x)$ be three real differentiable and monotonic functions in the interval $[a,b]$ and $|g(x)| \ll M$.*

(i) *If* $|f'(x)| \gg m > 0$, *then*

$$\int_a^b g(x)\, e\,(f(x))\, dx \ll M/m.$$

(ii) *If* $|f''(x)| \gg r > 0$, *then*

$$\int_a^b g(x)\, e\,(f(x))\, dx \ll M/r^{1/2}.$$

Proof. See [**13**, Chapter 21].

Lemma 3.2. *For any natural number* $q = q_1 q_2$, $(q_1, q_2) = 1$ *and characters* $\chi_a \pmod q = \chi_{a_1} \pmod{q_1}$, $\chi_{a_2} \pmod{q_2}$, $\chi_b \pmod q = \chi_{b_1} \pmod{q_1} \chi_{b_2} \pmod{q_2}$, $\chi_c \pmod q = \chi_{c_1} \pmod{q_1}$, $\chi_{c_2} \pmod{q_2}$ *and* $f = f_1 q_2 + f_2 q_1$, *there is*:

a) $C\,(\chi_a, q, k_q, b, f) = C\,(\chi_{a_1}, q_1, k_{q_1}, b, f_1)\, C\,(\chi_{a_2}, q_2, k_{q_2}, b, f_2)$,

b) $Z\,(N, q, k_q, \chi_a, \chi_b, \chi_c) = Z\,(N, q_1, k_{q_1}, \chi_{a_1}, \chi_{b_1}, \chi_{c_1})\, Z\,(N, q_2, k_{q_2}, \chi_{a_2}, \chi_{b_2}, \chi_{c_2})$.

c) *If* χ modulo p^β *is a both nonprimitive and nonprincipal character, i.e.,* χ *is induced by* χ^* modulo p^α, $1 \le \alpha < \beta$, *then for* $(b, p) = 1$, $(a, p) = 1$ *and* $0 \le \gamma < \beta$, *we have*

$$C\,(\chi, p^\beta, p^\gamma, b, a) = 0.$$

Proof. Parts a) and b) are shown in the same way as Lemma 4.4 a and b in [**2**]. Part c) is Lemma 4.3 in [**2**].

Lemma 3.3. *Set* $(a, q) = 1$ *and* $(b, q) = 1$ *throughout the lemmata* a) *and* b).

a) *Let* χ *be a character* modulo q. *Then*

$$C\,(\chi, q, 1, b, a) \ll q^{1/2}.$$

b)

$$C\left(\chi_{q,0}, q, k_q, b, a\right) = \begin{cases} \mu(q/k_q)e(tba/k_q) & \text{if } (q/k_q, k_q) = 1,\ tq/k_q \equiv 1 \pmod{k_q}, \\ 0 & \text{otherwise.} \end{cases}$$

c) *Let there be given any three characters* χ_1, χ_2, χ_3, modulo k^2. *Then*

$$Z\left(N, k^2, k, \chi_1, \chi_2, \chi_3\right) \neq 0 \Longrightarrow \chi_1,\ \chi_2,\ \chi_3$$

are primitive characters modulo k^2.

d) *For any three primitive characters* χ_i modulo r_i, $1 \leq i \leq 3$ *with* $k^2 || r$ *where* $[r_1, r_2, r_3] = r$, $r|q$, *and the principal character* χ_0 modulo q *we have*:

$$Z\left(N, q, k, \chi_1\chi_0, \chi_2\chi_0, \chi_3\chi_0\right) \neq 0 \Longrightarrow k^2 || r_i, \quad 1 \leq i \leq 3.$$

e) *For any* χ_1, χ_2, χ_3 modulo k^2

$$Z\left(N, k^2, k, \chi_1, \chi_2, \chi_3\right) \ll k^{-2}.$$

Proof. Part a) is contained in Lemmas 5.1 and 5.2 in [**12**]. Part b) is shown in [**16**].

c) If any $\chi_i = \chi_0 \pmod{k^2}$, $1 \leq i \leq 3$, then the lemma follows from Lemma 3.3 b). If any of χ_i is a nonprimitive character modulo k^2 that is induced by a primitive character modulo k, then the lemma follows from Lemma 3.2 c).

d) Applying Lemma 3.2 b), we can write $Z(N, q, k, \dots) = Z(N, r', k, \dots)A(N, l, 1)$, where $(r', l) = 1$, $r|r'$, and every prime factor that divides r' also divides r. From Lemma 3.2 c), we see that $Z(N, r', k, \dots) = 0$, if $r' \neq r$. Using the notation introduced in (2.3) and again Lemma 3.2 b), we find $Z(N, r, k, \dots) = Z(N, s, 1, \dots)Z(N, k^2, k, \dots)$. Thus, the proof can focus on terms $Z(N, q, \dots)$ that can be written as $Z(N, q, k, \dots) = Z(N, s, 1, \dots)Z(N, k^2, k, \dots)A(N, l, 1)$, where $(r, l) = 1$ and $(s, k) = 1$. Now the statement of this lemma follows from Lemma 3.3 c).

e) We know from Lemma 3.3 c) that we only have to consider characters χ_i, $1 \le i \le 3$, that are primitive modulo k^2. We know from [**3**, Lemma 5.1 c], that for a primitive character χ_i modulo k^2, we have $\chi_i(1+\bar{b}sk) = e(c_i\bar{b}s/k)$, where $(k, c_i) = 1$ and $\bar{b}b \equiv 1 \pmod{k^2}$. By definition,

$$\begin{aligned} C\left(\chi_i, k^2, k, b_i, a\right) &= \sum_{s=1}^{k} \chi_i(b_i+sk)\, e\left(\frac{ab_i+aks}{k^2}\right) \\ &= \chi_i(b_i)\sum_{s=1}^{k} \chi_i(1+\bar{b}_i sk)\, e\left(\frac{ab_i+aks}{k^2}\right) \\ &= \chi_i(b_i)\sum_{s=1}^{k} e\left(\frac{sc_i\bar{b}_i}{k^2}\right) e\left(\frac{ab_i+aks}{k^2}\right). \end{aligned} \tag{3.1}$$

Inserting (3.1) in the definition of $Z(\dots)$, we find

$$\begin{aligned} &Z\left(N, k^2, k, \chi_1, \chi_2, \chi_3\right) \\ &= \frac{\prod_{i=1}^3 \chi_i(b_i)}{\phi^3(k^2)} \sum_{a=1}^{k^2}{}^{*} \prod_{i=1}^{3}\left(\sum_{s_i=1}^{k} e\left(\frac{s_ic_i\bar{b}_i}{k^2}\right) e\left(\frac{ab_i+aks_i}{k^2}\right)\right) e\left(\frac{-aN}{k^2}\right) \\ &= \frac{\prod_{i=1}^3 \chi_i(b_i)}{\phi^3(k^2)} \sum_{s_1}^{k}\sum_{s_2}^{k}\sum_{s_3}^{k} e\left(\frac{s_1c_1\bar{b}_1+s_2c_2\bar{b}_2+s_3c_3\bar{b}_3}{k^2}\right) \\ &\times \sum_{a=1}^{k^2}{}^{*} e\left(\frac{a(b_1+b_2+b_3-N+s_1k+s_2k+s_3k)}{k^2}\right). \end{aligned} \tag{3.2}$$

Using that $b_1+b_2+b_3-N = Mk$, $M \in \mathbf{Z}$, we can write the inner sum in (3.2) as:

$$\begin{aligned} &\sum_{a=1}^{k^2}{}^{*} e\left(\frac{ak(M+s_1+s_2+s_3)}{k^2}\right) \\ &\qquad = k\sum_{a=1}^{k-1} e\left(\frac{a(M+s_1+s_2+s_3)}{k}\right) \\ &\qquad = \begin{cases} k(k-1) & \text{if } M+s_1+s_2+s_3 \equiv 0 \pmod{k}, \\ -k & \text{else.} \end{cases} \end{aligned}$$

Obviously,

(3.3)

$$\sharp\{s_1, s_2, s_3 : 1 \le s_1, s_2, s_3 \le k,\ M + s_1 + s_2 + s_3 \equiv 0 \pmod{k}\} = k^2.$$

Thus, noting that $k/\phi(k) \le 2$, we obtain from (3.2) and (3.3):

$$Z\left(N, k^2, k, \chi_1, \chi_2, \chi_3\right) \ll k^{-6} k^4 = k^{-2}.$$

Lemma 3.4. *Let there be given primitive characters* $\chi_i \bmod r_i$, $i = 1, 2, 3$, *the principal character* $\chi_0 \bmod q$ *and* $r = [r_1, r_2, r_3]$.

a) *If* $(r, k) = 1$, *then*

$$\sum_{\substack{q \le P \\ r|q}} \left|Z\left(N, q, k_q, \chi_1\chi_0, \chi_2\chi_0, \chi_3\chi_0\right)\right| \ll r^{-1/2} L.$$

b) *If* $k^m || r$, $m \in \{1, 2\}$, *then*

$$\sum_{\substack{q \le P \\ r|q}} \left|Z\left(N, q, k_q, \chi_1\chi_0, \chi_2\chi_0, \chi_3\chi_0\right)\right| \ll s^{-1/2} k^{-2} L.$$

c) *If* $(r, k) = 1$, *then*

$$\sum_{\substack{q \le P \\ kr|q}} \left|Z\left(N, q, k_q, \chi_1\chi_0, \chi_2\chi_0, \chi_3\chi_0\right)\right| \ll r^{-1/2} k^{-2} L.$$

Proof. a) Let J denote the left-hand side in Lemma 3.4 a). Arguing as in the proof of Lemma 3.3 d), we see that we can focus on terms $Z(N, q, \dots)$ which can be written as follows

$$Z\left(N, q, k_q, \dots\right) = Z\left(N, r, 1, \dots\right) A\left(N, l, k_l\right),$$

where $(l, r) = 1$. Thus

$$J \ll \left|Z\left(N, r, 1, \dots\right)\right| \sum_{l \le P/r} \left|A\left(N, l, k_l\right)\right|. \tag{3.4}$$

From Lemma 3.3 a), we derive

$$|Z(N,r,1,\dots)| \ll r^{-3} r r^{3/2} L_2^3 = r^{-1/2} L_2^3. \tag{3.5}$$

Lemma 3.3 b) implies that $|A(N,l,k_l)| \le l\phi^{-3}(l)$. Thus

$$\sum_{l \le P/r} A(N,l,k_l) \ll 1. \tag{3.6}$$

Part a) follows from (3.4)–(3.6). For the proof of part b), we use the definition (2.3) and Lemma 3.2 b) to write

$$Z(N,r,k,\dots) = Z(N,s,1,\dots) Z(N,k^m,k,\dots). \tag{3.7}$$

As in (3.5), we use Lemma 3.3 a) to estimate $Z(N,s,1,\dots)$. In order to estimate $Z(N,k^m,k,\dots)$, for $m=1$, we use the fact that by definition $|C(\chi,k,k,b,a)| \le 1$ whereas for $m=2$ we use Lemma 3.3 e). Thus,

$$Z(N,s,1,\dots) Z(N,k^m,k,\dots) \ll s^{-1/2} k^{-2} L_2^3. \tag{3.8}$$

The lemma then follows from (3.4), (3.6), (3.7) and (3.8). For the proof of part c), we argue as in (3.4):

$$J \ll |Z(N,r,1,\dots)| \sum_{\substack{l \le P/r \\ k|l}} |A(N,l,k)|. \tag{3.9}$$

We see from Lemma 3.3 b) that

$$\sum_{\substack{l \le P/r \\ k|l}} |A(N,l,k)| \le \sum_{\substack{l \le P/r \\ k|l}} \frac{l}{l^3} L_2^3 \le k^{-2} L_2^3 \sum_{l \le P/rk} l^{-2} \ll k^{-2} L_2^3. \tag{3.10}$$

Part c) now follows from (3.5), (3.9) and (3.10).

Lemma 3.5. *There exists a positive number J such that:*

a)

$$\sum_{q=1}^{\infty} \frac{1}{\phi(k/k_q)^3} A(N,q,k_q) = \sigma(N,k).$$

b)

$$\sum_{q\geq P}\frac{1}{\phi\left(k/k_q\right)^3}\left|A\left(N,q,k_q\right)\right|\ll (Pk)^{-1}L^J.$$

Proof. The proof is nearly identical to the proof of Lemma 4.6 in [**2**]. Whereas in [**2**] the estimate $k/\phi(k)\ll k^{\varepsilon}$ is used, here the estimate $k/\phi(k)\ll \log\log k$ is applied.

4. Treatment of the major arcs. We first consider the set $E_1(k)$. If $k\nmid q$, we find

$$S\left(\frac{a}{q}+\lambda,b_i\right)=\sum_{g=1}^{q}{}^{*}\,e\left(\frac{ga}{q}\right)\sum_{\substack{N/4<n\leq N\\ n\equiv b_i \pmod k\\ n\equiv g \pmod q}}\Lambda(n)\,e(n\lambda)+O(L^2).$$

We shall introduce the Dirichlet characters ξ mod k and χ mod q and obtain

$$\begin{aligned}S\left(\frac{a}{q}+\lambda,b_i\right)=&\frac{1}{\phi(k)\phi(q)}\,C\left(\chi_0,q,1,b_i,a\right)T(\lambda)+\frac{1}{\phi(k)\phi(q)}\\&+\sum_{\xi \bmod k}\bar{\xi}(b_i)\sum_{\chi \bmod q}C\left(\overline{\chi},q,1,b_i,a\right)W(\lambda,\xi\chi)+O(L^2).\end{aligned}$$

In the sequel, we will neglect the error term $O(L^2)$. We will see that its contribution will be dominated by other, larger error terms. We obtain from (2.5):

$$R_1(N)=R_1^m(N)+R_1^e(N), \tag{4.1}$$

where

$$\begin{aligned}R_1^m(N)=&\sum_{\substack{q\leq P_1\\ k\nmid q}}\frac{1}{\phi^3(k)\phi^3(q)}\sum_{a=1}^{q}{}^{*}\prod_{i=1}^{3}C\left(\chi_0,q,1,b_i,a\right)e\left(-\frac{a}{q}\,N\right)\\&\times\int_{-1/qQ}^{1/qQ}T^3(\lambda)\,e\left(-N\lambda\right)d\lambda,\end{aligned}$$

(4.2)

$$\begin{aligned}
R_1^e(N) = &\sum_{\substack{q \le P_1 \\ k \nmid q}} \frac{1}{\phi^3(k)\phi^3(q)} \sideset{}{^*}\sum_{a=1}^{q} e\left(-\frac{a}{q}N\right) \\
&\times \int_{-1/qQ}^{1/qQ} \prod_{i=1}^{3} \left(\sum_{\xi \bmod k} \bar{\xi}(b_i) \sum_{\chi \bmod q} C\left(\overline{\chi}, q, 1, b_i, a\right) W(\lambda, \xi\chi) \right) \\
&\times e\left(-\lambda N\right) d\lambda \\
&+ \sum_{i=1}^{3} \sum_{\substack{q \le P_1 \\ k \nmid q}} \frac{1}{\phi^3(k)\phi^3(q)} \sideset{}{^*}\sum_{a=1}^{q} e\left(-\frac{a}{q}N\right) \\
&\times \int_{-1/qQ}^{1/qQ} \prod_{\substack{j=1 \\ j \ne i}}^{3} \left(\sum_{\xi \bmod k} \bar{\xi}(b_j) \sum_{\chi \bmod q} C\left(\overline{\chi}, q, 1, b_j, a\right) W(\lambda, \xi\chi) \right) \\
&\times C\left(\chi_0, q, 1, b_i, a\right) T(\lambda)\, e\,\left(-\lambda N\right) d\lambda \\
&+ \sum_{i=1}^{3} \sum_{\substack{q \le P_1 \\ k \nmid q}} \frac{1}{\phi^3(k)\phi^3(q)} \sideset{}{^*}\sum_{a=1}^{q} e\left(-\frac{a}{q}N\right) \\
&\times \int_{-1/qQ}^{1/qQ} \left(\sum_{\xi \bmod k} \bar{\xi}(b_i) \sum_{\chi \bmod q} C\left(\overline{\chi}, q, 1, b_i, a\right) W(\lambda, \xi\chi) \right) \\
&\times \prod_{\substack{j=1 \\ j \ne i}}^{3} C\left(\chi_0, q, 1, b_j, a\right) T^2(\lambda)\, e\left(-\lambda N\right) d\lambda \\
=: &\sum_1 + \sum_2 + \sum_3 .
\end{aligned}$$

We first evaluate the main term $R_1^m(N)$ using (3.6) with $r = 1$,

$$\begin{aligned}
R_1^m(N) = &\frac{1}{\phi^3(k)} \sum_{\substack{q \le P_1 \\ k \nmid q}} A\left(N, q, 1\right) \int_{-1/2}^{1/2} T(\lambda)^3\, e\left(-N\lambda\right) d\lambda \\
&+ O\left(\frac{1}{\phi^3(k)} \sum_{\substack{q \le P_1 \\ k \nmid q}} \left|A\left(N, q, 1\right)\right| \int_{1/qQ}^{1/2} \frac{1}{|\lambda|^3}\, d\lambda \right)
\end{aligned}$$

$$= \frac{1}{\phi^3(k)} \sum_{\substack{q \le P_1 \\ k \nmid q}} A\,(N,q,1)\,\frac{N^2}{32} + O\left(\frac{(P_1Q)^2}{\phi^3(k)}\right) \tag{4.3}$$

$$= \frac{1}{\phi^3(k)} \sum_{\substack{q \le P_1 \\ k \nmid q}} A(N,q,1)\,\frac{N^2}{32} + O\left(N^2 k^{-4} L^{-A}\right),$$

where we have used $T(\lambda) \ll 1/|\lambda|$ and

$$\int_{-1/2}^{1/2} T(\lambda)^3\, e(-N\lambda)\, d\lambda = \sum_{N/4<n_1<N/2}\ \sum_{N/4<n_2<3N/4-n_1} 1 = \frac{N^2}{32} + O(N). \tag{4.4}$$

In the sequel we will without further mention use the fact that, for any character χ induced by a primitive character χ^*, we have $W(\chi, \chi\xi) = W(\lambda, \chi^*\xi) + O(L^2)$. Using Lemma 3.4 a), we estimate $\sum_1$:

$$\left|\sum_1\right| \le \frac{1}{\phi^3(k)} \sum_{\substack{q \le P_1 \\ k \nmid q}} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} \sum_{\chi_3 \bmod q} \sum_{\xi_1 \bmod k} \sum_{\xi_2 \bmod k} \sum_{\xi_3 \bmod k}$$

$$\times \left|Z\,(N,q,1,\chi_1,\chi_2,\chi_3)\right| \int_{-1/qQ}^{1/qQ} \prod_{j=1}^{3} |W(\lambda, \chi_j\xi_j)|\, d\,\lambda$$

$$\le \frac{1}{\phi^3(k)} \sum_{\substack{r_1 \le P_1 \\ k \nmid r_1}} \sum_{\substack{r_2 \le P_1 \\ k \nmid r_2}} \sum_{\substack{r_3 \le P_1 \\ k \nmid r_3}} \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*} \sum_{\chi_3 \bmod r_3}{}^{*}$$

$$\times \sum_{\xi_1 \bmod k} \sum_{\xi_2 \bmod k} \sum_{\xi_3 \bmod k}$$

$$\times \int_{-1/[r_1,r_2,r_3]Q}^{1/[r_1,r_2,r_3]Q} \prod_{j=1}^{3} \left(|W(\lambda, \chi_j\xi_j)| + L^2\right) d\lambda \tag{4.5}$$

$$\times \sum_{\substack{q \le P_1 \\ [r_1,r_2,r_3] \mid q}} \left|Z\,(N,q,1,\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0)\right|$$

$$\ll \frac{L}{\phi^3(k)} \sum_{\substack{r_1 \le P_1 \\ k \nmid r_1}} \sum_{\substack{r_2 \le P_1 \\ k \nmid r_2}} \sum_{\substack{r_3 \le P_1 \\ k \nmid r_3}} [r_1, r_2, r_3]^{-1/2}$$

$$\times \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*} \sum_{\chi_3 \bmod r_3}{}^{*} \sum_{\xi_1 \bmod k} \sum_{\xi_2 \bmod k} \sum_{\xi_3 \bmod k}$$

$$\times \int_{-1/[r_1,r_2,r_3]Q}^{1/[r_1,r_2,r_3]Q} \prod_{j=1}^{3} \left(|W(\lambda, \chi_j \xi_j)| + L^2\right) d\,\lambda,$$

In the following, we will neglect the error terms L^2 in the last integral in (4.5) as their contribution will be dominated by other terms. As a character ξ modulo k is either primitive or the principal character modulo k, the following relation holds for all characters χ_i and ξ_i, $i \in \{1, 2, 3\}$, over which is summed in (4.5):

$$(\chi\xi)^* = \begin{cases} \chi^* & \text{if } \xi = \xi_0, \\ \chi^*\xi & \text{otherwise.} \end{cases} \tag{4.6}$$

Thus we see from (4.5) and (4.6) and by the notation for s_i introduced in (2.2),

(4.7)

$$\sum_1 \ll k^{-3} L^2 \Bigg(\sum_{\substack{r_1 \le P_1 k \\ k||r_1}} \sum_{\substack{r_2 \le P_1 k \\ k||r_2}} \sum_{\substack{r_3 \le P_1 k \\ k||r_3}} + \sum_{\substack{r_1 \le P_1 k \\ k||r_1}} \sum_{\substack{r_2 \le P_1 k \\ k||r_2}} \sum_{\substack{r_3 \le P_1 \\ k\nmid r_3}}$$

$$+ \sum_{\substack{r_1 \le P_1 k \\ k||r_1}} \sum_{\substack{r_2 \le P_1 \\ k\nmid r_2}} \sum_{\substack{r_3 \le P_1 \\ k\nmid r_3}} + \sum_{\substack{r_1 \le P_1 \\ k\nmid r_1}} \sum_{\substack{r_2 \le P_1 \\ k\nmid r_2}} \sum_{\substack{r_3 \le P_1 \\ k\nmid r_3}} \Bigg) [s_1, s_2, s_3]^{-1/2}$$

$$\times \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*} \sum_{\chi_3 \bmod r_3}{}^{*} \int_{-1/[s_1,s_2,s_3]Q}^{1/[s_1,s_2,s_3]Q} \prod_{j=1}^{3} |W(\lambda, \chi_j)| \, d\lambda$$

$$=: \sum_{i=1}^{4} \sum_{1,i},$$

where each $\sum_{1,i}$ stands for one of the multiple sums in (4.7). Using

$[s_1, s_2, s_3]^{1/2} \geq s_2^{1/4} s_3^{1/4}$, we obtain

$$
\begin{aligned}
\sum_{1,1} &\ll k^{-3} L^2 \sum_{\substack{r_1 \leq P_1 k \\ k | r_1}} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \max_{|\lambda| \leq 1/s_1 Q} |W(\lambda, \chi_1)| \\
&\quad \times \sum_{\substack{r_2 \leq P_1 k \\ k | r_2}} s_2^{-1/4} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \left(\int_{-1/s_2 Q}^{1/s_2 Q} |W(\lambda, \chi_2)|^2 \, d\lambda \right)^{1/2} \\
&\quad + \sum_{\substack{r_3 \leq P_1 k \\ k | r_3}} s_3^{-1/4} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \left(\int_{-1/s_3 Q}^{1/s_3 Q} |W(\lambda, \chi_3)|^2 \, d\,\lambda \right)^{1/2} \\
&=: k^{-2} L^2 I_A W_A^2,
\end{aligned}
\tag{4.8}
$$

where

$$
I_A = k^{-1/3} \sum_{\substack{r \leq P_1 k \\ k | r}} \sideset{}{^*}\sum_{\chi \pmod r} \max_{|\lambda| \leq k/rQ} |W(\lambda, \chi)|,
$$

$$
W_A = k^{-1/3} \sum_{\substack{r \leq P_1 k \\ k | r}} \left(\frac{r}{k} \right)^{-1/4} \sideset{}{^*}\sum_{\chi \pmod r} \left(\int_{-k/rQ}^{k/rQ} |W(\lambda, \chi)|^2 \, d\,\lambda \right)^{1/2}.
$$

Arguing similarly, we obtain

$$
\sum_{i=2}^{4} \sum_{1,i} \ll k^{-2} L^2 \left(I_A W_A W_B + I_A W_B^2 + I_B W_B^2 \right), \tag{4.9}
$$

where

$$
I_B = k^{-1/3} \sum_{\substack{r \leq P_1 \\ k \nmid r}} \sideset{}{^*}\sum_{\chi \pmod r} \max_{|\lambda| \leq 1/rQ} |W(\lambda, \chi)|,
$$

$$
W_B = k^{-1/3} \sum_{\substack{r \leq P_1 \\ k \nmid r}} r^{-1/4} \sideset{}{^*}\sum_{\chi \pmod r} \left(\int_{-1/rQ}^{1/rQ} |W(\lambda, \chi)|^2 \, d\,l \right)^{1/2}.
$$

In the same way we find

(4.10)
$$\sum_2 + \sum_3 \ll k^{-2} L^2 \max_{|\lambda|\le 1/Q} |T(\lambda)| \left(W_B^2 + W_B W_A + W_A^2\right) + k^{-2} L^2 \max_{|\lambda|\le 1/Q} |T(\lambda)| \left(\int_{-1/Q}^{1/Q} |T(\lambda)|^2 \, dl\right)^{1/2} (W_B + W_A)\,.$$

We have

$$\max_{|\lambda|\le 1/Q} |T(\lambda)| \ll N. \tag{4.11}$$

Using $T(\lambda) \le \min(N, (1/\lambda))$, we see that

$$\left(\int_{-1/Q}^{1/Q} |T(\lambda)|^2 \, dl\right)^{1/2} \ll N^{1/2}. \tag{4.12}$$

Therefore, we see from (4.2) and (4.7)–(4.12):

$$\begin{aligned} R_1^e(N) \ll k^{-2} L^2 \Big(& N \left(W_B^2 + W_B W_A + W_A^2\right) + N^{3/2}(W_B + W_A) \\ & + I_A W_A^2 + I_A W_A W_B + I_A W_B^2 + I_B W_B^2 \Big). \end{aligned} \tag{4.13}$$

For $q \in E_2(k)$, we see

$$\begin{aligned} S\left(\frac{a}{q} + \lambda, b_i\right) &= \sideset{}{^*}\sum_{\substack{g=1 \\ g\equiv b_i \pmod k}}^{q} e\left(\frac{ga}{q}\right) \sum_{\substack{N/4<n\le N \\ n\equiv b_i \pmod k \\ n\equiv g \pmod q}} \Lambda(n)\, e(n\lambda) \\ &= \sideset{}{^*}\sum_{\substack{g=1 \\ g\equiv b_i \pmod k}}^{q} e\left(\frac{ga}{q}\right) \sum_{\substack{N/4<n\le N \\ n\equiv g \pmod q}} \Lambda(n)\, e(n\lambda) \\ &= \frac{1}{\phi(q)}\, C(\chi_0, q, k, b_i, a)\, T(\lambda) \\ &\quad + \frac{1}{\phi(q)} \sum_{\chi \bmod q} C(\overline{\chi}, q, k, b_i, a) W(\lambda, \chi). \end{aligned}$$

Arguing as in (4.1)–(4.3), we obtain by applying (3.6) in the same way as in (4.3) and using (4.4):

$$R_2(N) = R_2^m(N) + R_2^e(N), \tag{4.14}$$

where

$$R_2^m(N) = \sum_{\substack{q\le P_2\\ k|q}} A\left(N,q,k\right)\frac{N^2}{32} + O\left(N^2k^{-3}L^{-A}\right), \tag{4.15}$$

$$\begin{aligned}
R_2^e(N) = &\sum_{\substack{q\le P_2\\ k|q}} \frac{1}{\phi^3(q)} \sum_{a=1}^{q}{}^{*} \int_{-1/qQ}^{1/qQ} \\
&\times \prod_{i=1}^{3}\left(\sum_{\chi \bmod q} C\left(\overline{\chi},q,k,b_i,a\right)W(\lambda,\chi)\right) e\left(-\frac{a}{q}N-\lambda N\right)d\lambda \\
&+\sum_{i=1}^{3}\sum_{\substack{q\le P_2\\ k|q}} \frac{1}{\phi^3(q)} \sum_{a=1}^{q}{}^{*} \int_{-1/qQ}^{1/qQ} \\
&\times \prod_{\substack{j=1\\ j\ne i}}^{3}\left(\sum_{\chi \bmod q} C\left(\overline{\chi},q,k,b_j,a\right)W(\lambda,\chi)\right) \\
&\times C\left(\chi_0,q,k,b_i,a\right)T(\lambda)\, e\left(-\frac{a}{q}N-\lambda N\right)d\lambda \\
&+\sum_{i=1}^{3}\sum_{\substack{q\le P_2\\ k|q}} \frac{1}{\phi^3(q)} \sum_{a=1}^{q}{}^{*} \int_{-1/qQ}^{1/qQ} \sum_{\chi \bmod q} C\left(\overline{\chi},q,k,b_i,a\right)W(\lambda,\chi) \\
&\times \prod_{\substack{j=1\\ j\ne i}}^{3} C\left(\chi_0,q,k,b_j,a\right)T^2(\lambda)\, e\left(-\frac{a}{q}N-\lambda N\right)d\lambda \\
=: &\sum_4+\sum_5+\sum_6 .
\end{aligned} \tag{4.16}$$

Arguing similarly as in (4.5) and using Lemma 3.3 d), we see

$$\begin{aligned}\sum_4 &= \sum_{\substack{q\le P_2\\ k|q}} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} \sum_{\chi_3 \bmod q} |Z(N,q,k,\chi_1,\chi_2,\chi_3)| \\ &\quad \times \int_{-1/qQ}^{1/qQ} \prod_{j=1}^{3} |W(\lambda,\chi_j)|\, d\lambda \end{aligned} \tag{4.17}$$

$$\begin{aligned} &\ll \Bigg(\sum_{\substack{r_1\le P_2\\ k||r_1}} \sum_{\substack{r_2\le P_2\\ k||r_2}} \sum_{\substack{r_3\le P_2\\ k||r_3}} + \sum_{\substack{r_1\le P_2\\ k||r_1}} \sum_{\substack{r_2\le P_2\\ k||r_2}} \sum_{\substack{r_3\le P_2/k\\ k\nmid r_3}} \\ &\quad + \sum_{\substack{r_1\le P_2\\ k||r_1}} \sum_{\substack{r_2\le P_2/k\\ k\nmid r_2}} \sum_{\substack{r_3\le P_2/k\\ k\nmid r_3}} + \sum_{\substack{r_1\le P_2/k\\ k\nmid r_1}} \sum_{\substack{r_2\le P_2/k\\ k\nmid r_2}} \sum_{\substack{r_3\le P_2/k\\ k\nmid r_3}} \\ &\quad + \sum_{\substack{r_1\le P_2\\ k^2||r_1}} \sum_{\substack{r_2\le P_2\\ k^2||r_2}} \sum_{\substack{r_3\le P_2\\ k^2||r_3}} \Bigg) \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \\ &\quad \times \int_{-1/[r_1,r_2,r_3]Q}^{1/[r_1,r_2,r_3]Q} \prod_{j=1}^{3} \left(|W(\lambda,\chi_j)| + L^2\right) d\lambda \\ &\quad \times \sum_{\substack{q\le P_2\\ [r_1,r_2,r_3]|q\\ k|q}} |Z(N,q,k_q,\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0)| \\ &=: \sum_{i=1}^{5} \sum_{4,i}, \end{aligned} \tag{4.18}$$

where each $\sum_{4,i}$ stands for one of the multiple sums in (4.17). The condition $k|q$ in the index of the sum $\sum_{\substack{q\le P_2\\ [r_1,r_2,r_3]||q\\ k|q}} |Z(N,q,k_q,\chi_1\chi_0,$ $\chi_2\chi_0,\chi_3\chi_0)$ is only necessary for the sum $\sum_{4,4}$. In the other cases, $k|[r_1,r_2,r_3]$ which implies $k|q$. Thus, we will only make use of the condition when we estimate the sum $\sum_{4,4}$. Again, we neglect the error terms L^2 in the last expression as they will be dominated in the sequel by other error terms. In order to estimate the $\sum_{4,1}$, we use the fact that for all q considered in (4.17), there holds $k^3 \nmid q$ because of $q \le P_2$.

This allows us to apply Lemma 3.4 b). Using (2.2), Lemma 3.4 b) and the relation $[s_1, s_2, s_3]^{1/2} \geq s_1^{1/6} s_2^{1/6} s_3^{1/6}$, we argue as in (4.8):

(4.19)

$$\sum_{4,1} \ll k^{-2} L \sum_{\substack{r_1 \leq P_2 \\ k || r_1}} \sum_{\substack{r_2 \leq P_2 \\ k || r_2}} \sum_{\substack{r_3 \leq P_2 \\ k \, || r_3}} [s_1, s_2, s_3]^{-1/2}$$

$$+ \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*} \sum_{\chi_3 \bmod r_3}{}^{*} \int_{-1/[r_1, r_2, r_3]Q}^{1/[r_1, r_2, r_3]Q} \prod_{j=1}^{3} |W(\lambda, \chi_j)| \, d\lambda$$

$$\ll k^{-2} L I_C W_C^2,$$

where

$$I_C = \sum_{\substack{r \leq P_2 \\ k | r}} \left(\frac{r}{k} \right)^{-1/6} \sum_{\chi \pmod r}{}^{*} \max_{|\lambda| \leq 1/rQ} |W(\lambda, \chi)|,$$

$$W_C = \sum_{\substack{r \leq P_2 \\ k | r}} \left(\frac{r}{k} \right)^{-1/6} \sum_{\chi \pmod r}{}^{*} \left(\int_{-1/rQ}^{1/rQ} |W(\lambda, \chi)|^2 \, d\lambda \right)^{1/2}.$$

Arguing as in (4.10), using Lemma 3.4 b) and the relation $[s_1, s_2, s_3]^{1/2}$ $\geq s_1^{1/6} s_2^{1/6} s_3^{1/6}$, we obtain

$$\sum_{4,2} + \sum_{4,3} \leq k^{-2} L \left(I_D W_C^2 + I_D W_D W_C \right), \tag{4.20}$$

where

$$I_D = \sum_{\substack{r \leq P_2/k \\ k \nmid r}} \sum_{\chi \pmod r}{}^{*} \max_{|\lambda| \leq 1/rQ} |W(\lambda, \chi)|,$$

$$W_D = \sum_{\substack{r \leq P_2/k \\ k \nmid r}} r^{-1/4} \sum_{\chi \pmod r}{}^{*} \left(\int_{-1/rQ}^{1/rQ} |W(\lambda, \chi)|^2 \, d\lambda \right)^{1/2}.$$

For the estimate of $\sum_{4,4}$, we argue as in (4.19) and apply Lemma 3.4 c):

(4.21)

$$\sum_{4,4} \ll k^{-2}L \sum_{\substack{r_1\le P_2/k\\ k\nmid r_1}} \sum_{\substack{r_2\le P_2/k\\ k\nmid r_2}} \sum_{\substack{r_3\le P_2/k\\ k\nmid r_3}} [r_1,r_2,r_3]^{-1/2}$$

$$+ \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*} \sum_{\chi_3 \bmod r_3}{}^{*} \int_{-1/[r_1,r_2,r_3]Q}^{1/[r_1,r_2,r_3]Q} \prod_{j=1}^{3} |W(\lambda,\chi_j)|\, d\lambda$$

$$\le k^{-2}LI_D W_D^2.$$

As $k^3 \nmid q$ for all considered q, we use Lemma 3.4 b) to estimate the sum $\sum_{4,5}$:

(4.22)

$$\sum_{4,5} \ll k^{-2}L \sum_{\substack{r_1\le P_2\\ k^2||r_1}} \sum_{\substack{r_2\le P_2\\ k^2||r_2}} \sum_{\substack{r_3\le P_2\\ k^2||r_3}} [s_1,s_2,s_3]^{-1/2}$$

$$+ \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*} \sum_{\chi_3 \bmod r_3}{}^{*} \int_{-1/[r_1,r_2,r_3]Q}^{1/[r_1,r_2,r_3]Q} \prod_{j=1}^{3} |W(\lambda,\chi_j)|\, d\lambda$$

$$\le k^{-2}LI_E W_E^2,$$

where

$$I_E = \sum_{\substack{r\le P_2\\ k^2|r}} \left(\frac{r}{k^2}\right)^{-1/6} \sum_{\chi \pmod r}{}^{*} \max_{|\lambda|\le 1/rQ} |W(\lambda,\chi)|,$$

$$W_E = \sum_{\substack{r\le P_2\\ k^2|r}} \left(\frac{r}{k^2}\right)^{-1/6} \sum_{\chi \pmod r}{}^{*} \left(\int_{-1/rQ}^{1/rQ} |W(\lambda,\chi)|^2\, d\lambda\right)^{1/2}.$$

Arguing as in (4.9), we obtain

(4.23)

$$\sum_5 + \sum_6 \ll k^{-2}L\Bigg(\max_{|l|\le 1/Q} |T(\lambda)| \left(W_C^2 + W_CW_D + W_D^2 + W_E^2\right)$$

$$+ \max_{|\lambda|\le 1/Q} |T(\lambda)| \left(\int_{-1/Q}^{1/Q} |T(\lambda)|^2\, dl\right)^{1/2} (W_C + W_D + W_E)\Bigg).$$

Therefore, we see from (4.11), (4.12), and (4.16)–(4.23):

$$\begin{aligned} R_2^e(N) \ll k^{-2} L\Big(& N\left(W_C^2 + W_D W_C + W_D^2 + W_E^2\right) \\ & + N^{3/2}(W_C + W_D + W_E) \\ & + I_C W_C^2 + I_D W_C^2 + I_D W_D W_C + I_D W_D^2 + I_E W_E^2\Big). \end{aligned} \tag{4.24}$$

Using Lemma 3.5, we see from (4.3) and (4.15) that for a sufficiently large $G = G(A)$

$$R_1^m(N) + R_2^m(N) = \sigma(N,k)\,\frac{N^2}{32} + O\left(N^2 k^{-2} L^{-A}\right). \tag{4.25}$$

Thus we see from (4.1), (4.13), (4.14), (4.24) and (4.25) that the proof of (2.7) reduces to the proof of the following two lemmas:

Lemma 4.1. *If $k \le N^{(2/15)-\varepsilon}$, then for $F \in \{A, B, D\}$*

$$W_F \ll N^{1/2} L^{-A}$$

for any $A > 0$.

For $k \le N^{(5/48)-\varepsilon}$ and if none of the integers $q \in A_k$ is N-exceptional, then for $F \in \{C, E\}$

$$W_F \ll N^{1/2} L^{-A}$$

for any $A > 0$.

Lemma 4.2. *If $k \le N^{(2/15)-\varepsilon}$, then for $F \in \{A, B, C, D, E\}$*

$$I_F \ll N L^M$$

for a certain $M > 0$.

In the sequel, we will also use the following lemma, which is the estimate (1.1) in [**6**]:

Lemma 4.3. *Let $N^*(\alpha, T, q)$ denote the number of zeros $\sigma + it$ of all $L-$functions to primitive characters modulo q within the region $\sigma \geq \alpha$, $|t| \leq T$. Then, for a positive integer m,*

$$\sum_{\substack{q\leq P\\ m|q}} N^*(\alpha, T, q) \ll \left(\frac{P^2T}{m}\right)^{((12/5)+\varepsilon)(1-\alpha)}.$$

5. Proof of Lemma 4.1 for W_A. In order to prove the lemma it is enough to show that

$$W_{A,R} \ll N^{1/2}\left(\frac{R}{k}\right)^{1/4} k^{1/3}L^{-A}, \tag{5.1}$$

where

$$W_{A,R} = \sum_{\substack{r\sim R\\ k|r}} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ r)} \left(\int_{-k/rQ}^{k/rQ} |W(\lambda,\chi)|^2\, d\,\lambda\right)^{1/2}$$

for $R \leq P_1 k/2$. Applying Lemma 1, [**4**], we see

$$\int_{-k/rQ}^{k/rQ} |W(\lambda,\chi)|^2\, d\,\lambda \tag{5.2}$$

$$\ll (QR/k)^{-2}\int_{N/8}^{N}\left|\sum_{\substack{t<m\leq t+Qr/k\\ N/4<m\leq N}} \Lambda(m)\,\chi(m) - E_0(\chi)\sum_{\substack{t<m\leq t+Qr/k\\ N/4<m\leq N}} 1\right|^2 dt.$$

We note that $E_0(\chi) = 0$ because of $R \geq k$ and the primitivity of the characters. We set $X = \max(N/4, t)$ and $X + Y = \min(N, t + Qr/k)$. We apply a slight modification of Heath-Brown's identity [**5**]

$$-\frac{\zeta'}{\zeta}(s) = \sum_{j=1}^{K}\binom{K}{j}(-1)^{j-1}\zeta'(s)\zeta^{j-1}(s)M^j(s) - \frac{\zeta'}{\zeta}(s)\left(1-\zeta(s)M(s)\right)^K,$$

with $K = 5$ and

$$M(s) = \sum_{n\leq N^{1/5}} \mu(n)n^{-s}$$

to the sum

$$\sum_{X<m\le X+Y} \Lambda(m)\,\chi(m).$$

Arguing exactly as in part III, [**19**] we find by applying Heath-Brown's identity and Perron's summation formula that the inner sum of (5.2) is a linear combination of $O(L^c)$ terms of the form

$$\begin{aligned} &S_{I_{a_1},\dots,I_{a_{10}}} \\ &\quad = \frac{1}{2\pi i}\int_{-T}^{T} F\left(\frac{1}{2}+iu,\chi\right)\frac{(X+Y)^{((1/2)+iu)}-X^{((1/2)+iu)}}{(1/2)+iu}\,du \\ &\quad\quad + O(T^{-1}NL^2), \end{aligned}$$

where $2\le T\le N$,

$$F(s,\chi)=\prod_{j=1}^{10} f_j(s,\chi), \qquad f_j(s,\chi)=\sum_{n\in I_j} a_j(n)\,\chi(n)n^{-s},$$

$$a_j(n)=\begin{cases} \log n \text{ or } 1 & j=1, \\ 1 & 1<j\le 5, \\ \mu(n) & 6\le j\le 10. \end{cases} \qquad I_j=(N_j,\,2N_j], \quad 1\le j\le 10,$$

(5.3) $$N \ll \prod_{j=1}^{10} N_j \ll N, \quad N_j\le N^{1/5}, \quad 6\le j\le 10.$$

Since

$$\frac{(X+Y)^{((1/2)+iu)}-X^{((1/2)+iu)}}{(1/2)+iu} \ll \min\Big(QRk^{-1}N^{-1/2}, N^{1/2}\,(|u|+1)^{-1}\Big)$$

by taking $T=N$ and $T_0=N(QR/k)^{-1}$, we conclude that, for a sufficiently large $G=G(M)$, $S_{I_{a_1},\dots,I_{a_{10}}}$ is bounded by

$$\begin{aligned} &\ll QRk^{-1}N^{-1/2}\int_{-T_0}^{T_0}\left|F\left(\frac{1}{2}+iu,\chi\right)\right|du \\ &\quad + N^{1/2}\int_{T_0\le|u|\le T}\left|F\left(\frac{1}{2}+iu,\chi\right)\right|\frac{du}{|u|}+L^2, \end{aligned}$$

Thus we derive from (5.2) that, in order to prove (5.1), it is enough to show that, for $R \le P_1 k/2$,

$$\sum_{\substack{r\sim R\\ k|r}} \sideset{}{^*}\sum_{\chi} \int_0^{T_0} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}R^{1/4}k^{1/12}L^{-A}, \tag{5.4}$$

(5.5)
$$\sum_{\substack{r\sim R\\ k|r}} \sideset{}{^*}\sum_{\chi} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{-1/2}QR^{5/4}k^{-1/12}T_1L^{-A},$$

$$T_0 < |T_1| \le T.$$

The inequalities (5.4) and (5.5) are both derived from the following lemma which is shown for $m = 1$ in Lemma 5.2, [**10**] and for the general case $m \ge 1$ in Lemma 2.1 in [**8**].

Lemma 5.1. *Let $F(s,\chi)$ be defined as above. Then, for any $R \ge 1$ and $T_2 > 0$,*

$$\sum_{\substack{r\sim R\\ m|r}} \sideset{}{^*}\sum_{\chi} \int_{T_2}^{2T_2} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \tag{5.6}$$
$$\ll \left(\frac{R^2}{m}\,T_2 + \frac{R}{m^{1/2}}\,T_2^{1/2}N^{3/10} + N^{1/2}\right)L^c.$$

Using (2.1) and (2.4), the estimates (5.4) and (5.5) follow from Lemma 5.1 by setting $T_2 = T_0$ and $T_2 = T_1$, respectively, provided that $k \le N^{2/15-\varepsilon}$ and H is chosen sufficiently large in (2.1).

6. Proof of Lemma 4.2 for I_A. To prove the lemma it is enough to show that

$$\max_{R\le P_1k/2} \sum_{\substack{r\sim R\\ k|r}} \sideset{}{^*}\sum_{\chi \pmod r} \max_{|\lambda|\le k/rQ} |W(\lambda,\chi)| \ll Nk^{1/3}L^c.$$

Arguing as in the section before (we do not have to apply Gallagher's lemma here) we find

$$W(\lambda,\chi) \ll L^c \max_{I_{a_1},\dots,I_{a_{10}}} \left| \int_{-T}^{T} F\left(\frac{1}{2}+it,\chi\right) dt \int_{N/4}^{N} u^{-1/2} e\left(\frac{t}{2\pi}\log u + \lambda u\right) du \right| + L^2 k^3.$$

Here, we have set $T = N$ and used that $|\lambda| \le k/Q$. Estimating the inner integral by Lemma 3.1 we obtain

$$\left| \int_{N/4}^{N} u^{-1/2} e\left(\frac{t}{2\pi}\log u + lu\right) du \right| \ll N^{-1/2} \min\left(\frac{N}{\sqrt{|t|+1}}, \frac{N}{\min_{N/2<u\le N} |t+2\pi\lambda u|} \right).$$

Taking $T_0 = 4\pi N(QR/k)^{-1}$ we conclude that in order to prove the lemma it is enough to prove that

$$\sum_{\substack{r\sim R\\ k|r}} \sum_{\chi}{}^{*} \int_0^{T_0} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2} T_0^{1/2} k^{1/3} L^c,$$

$$\sum_{\substack{r\sim R\\ k|r}} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2} k^{1/3} T_1 L^c, \quad T_0 < |T_1| \le T.$$

These estimates follow from Lemma 5.1 for $k \le N^{2/15-\varepsilon}$.

7. Proof of Lemma 4.1 for $\mathbf{W}_B$, $\mathbf{W}_C$, $\mathbf{W}_D$ and $\mathbf{W}_E$. Arguing analogously to Section 5, we find that the proof of Lemma 4.1 for $F = B$ reduces to the proof of the following two estimates: For $T = N$, $T_0 = N(QR)^{-1}$, $R \le P_1/2$ and $k \le N^{3/16-\varepsilon}$, there must hold

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_0^{T_0} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2} R^{1/4} k^{1/3} L^{-A}, \tag{7.1}$$

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{-1/2} Q R^{5/4} k^{1/3} T_1 L^{-A}, \quad T_0 < |T_1| \le T. \tag{7.2}$$

The estimates (7.1) and (7.2) follow from (2.1) and Lemma 5.1. For the case $F = C$, we treat separately the cases $R/k \le L^V$ and $R/k \ge L^V$ for a sufficiently large V to be determined later. In the second case, it is enough to show, using Lemma 5.1, that for $T = N$, $T_0 = N(QR)^{-1}$, $R \le P_2/2$, and $k \le N^{3/20-\varepsilon}$, we have

$$\sum_{\substack{r\sim R\\ k|r}} \sum_{\chi}{}^{*} \int_0^{T_0} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}\left(\frac{R}{k}\right)^{1/6} L^{-A}, \tag{7.3}$$

(7.4)

$$\sum_{\substack{r\sim R\\ k|r}} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{-1/2} Q R^{7/6} k^{-1/6} T_1 L^{-A},$$

$$T_0 < |T_1| \le T.$$

In the case $R/k \le L^V$, we can estimate the sum on the righthand side of (5.2) by using the zero expansion of the von Mangoldt-function:

$$\begin{aligned}
&\sum_{\substack{t<m\le t+Qr\\ N/4<m\le N}} \Lambda(m)\,\chi(m) - E_0(\chi) \sum_{\substack{t<m\le t+Qr\\ N/4<m\le N}} 1\\
&\quad = \sum_{X<m\le X+Y} \Lambda(m)\,\chi(m) - E_0(\chi) \sum_{X<m\le X+Y} 1\\
&\quad \ll \sum_{|\mathrm{Im}\,\rho|\le T_3} \left| \frac{(X+Y)^\rho}{\rho} - \frac{X^\rho}{\rho}\right| + O\left(\frac{N}{T_3}\,L^2\right)\\
&\quad \ll QR \sum_{|\mathrm{Im}\,\rho|\le T_3} N^{\beta-1} + O\left(\frac{N}{T_3}\,L^2\right),
\end{aligned} \tag{7.5}$$

where ρ runs over the nontrivial zeros of the L-function corresponding to $\chi \bmod r$ with $|\mathrm{Im}\,\rho| \le T_3$ and $\beta = \mathrm{Re}\,\rho$. Arguing as in (5.2), we see from (7.5) for $T_3 = k^2 L^{2V}$ that

$$\begin{aligned}
&\int_{1/rQ}^{1/rQ} |W(\lambda,\chi)|^2 \, d\lambda\\
&\qquad \ll N\left(\sum_{|\mathrm{Im}\,\rho|\le k^2L^{2V}} N^{\beta-1}\right)^2 + O\left((Qr)^{-2} N^3 k^{-4} L^{4-4V}\right).
\end{aligned}$$

Using (1.2) and defining $W_{C,R}$ analogously to (5.1), we use the assumptions of Theorem 2 and Lemma 4.3 and obtain for $k \le N^{5/36-\varepsilon}$
(7.6)

$$\begin{aligned} W_{C,R} &\ll N^{1/2} \sum_{\substack{r \le kL^V \\ k|r}} \sum_{\chi \bmod r}{}^{*} \sum_{|\operatorname{Im}\rho| \le k^2 L^{2V}} N^{\beta-1} + N^{1/2}L^{-A} \\ &\ll N^{1/2}L^C \max_{1/2 \le \beta \le 1 - EL_2/L} \left(N^{((5/12)-2\varepsilon)((12/5)+\varepsilon)(1-\beta)} N^{\beta-1} \right) \\ &\quad + N^{1/2}L^{-A} \\ &\ll N^{1/2}L^{-A}, \end{aligned}$$

for a sufficiently large $E = E(A, \varepsilon)$. In the case $F = D$, we distinguish between the cases $R > L^W$ for a sufficiently large W to be determined later and $R \le L^W$. In the first case, we argue as in Section 4 and see that it is enough to show, using Lemma 5.1, the following. If $T = N$ and $T_0 = N(QR)^{-1}$, $r \le P_2/2k$ and $k \le N^{4/25-\varepsilon}$, then:

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_0^{T_0} \left| F\left(\frac{1}{2} + it, \chi\right) \right| dt \ll N^{1/2}R^{1/4}L^{-A},$$

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2} + it, \chi\right) \right| dt \ll N^{-1/2}QR^{5/4}T_1L^{-A},$$

$$T_0 < |T_1| \le T.$$

If $R \le L^W$, we apply Lemma 4.3 and the fact that $L(\sigma + it, \chi)$ with χ mod r and $r \le L^D$ has no zeros in the region, see [**15**, VIII Satz 6.2)

$$\sigma \ge 1 - \delta(T) := 1 - \frac{c_0}{\log r + (\log(T+2))^{4/5}}, \quad |t| \le T,$$

where c_0 is an absolute constant. Taking $T = N^{1/3}$ and $k \le N^{3/20-\varepsilon}$, we obtain from Lemma 4.3 from (7.5)

$$\begin{aligned} \int_{-1/Qr}^{1/Qr} |W(\lambda,\chi)|^2 \, d\lambda &\ll N\Bigg(\sum_{|\operatorname{Im}\rho| \le N^{1/3}} N^{\beta-1} \Bigg)^2 + (Qr)^{-2}N^{1+(4/3)}L^4 \\ &\ll NL^c \Bigg(\max_{(1/2) \le \beta \le 1-\delta(T)} N^{((4/5)+\varepsilon)(1-\beta)} N^{(\beta-1)} \Bigg)^2 + N^{1/3}k^4L^{2W+4} \\ &\ll N \exp(-cL^{1/5}). \end{aligned}$$

This proves the lemma for $R \le L^W$. For the case $F = E$, we treat separately the cases $R/k^2 \le L^V$ and $R/k^2 \ge L^V$ for a sufficiently large V to be determined later. In the second case, it is enough to show, using Lemma 5.1, that for $T = N$, $T_0 = N(QR)^{-1}$, $R \le P_2/2$ and $k \le N^{5/48-\varepsilon}$, we have

$$\sum_{\substack{r\sim R\\ k^2|r}} \sum_{\chi}{}^{*} \int_0^{T_0} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}\left(\frac{R}{k^2}\right)^{1/6} L^{-A},$$

$$\sum_{\substack{r\sim R\\ k^2|r}} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{-1/2}QR^{7/6}k^{-1/3}T_1L^{-A},$$

$$T_0 < |T_1| \le T.$$

For $R/k^2 \le L^V$, we argue as in (7.6).

8. Proof of Lemma 4.2 for $\mathbf{I}_B$, $\mathbf{I}_C$, $\mathbf{I}_D$, and $\mathbf{I}_E$. Throughout this section we set $T = N$ and $T_0 = N(QR)^{-1}$. Arguing as in Section 6, we see that to estimate I_B it is enough to show that for $k \le N^{3/16-\varepsilon}$ and $R \le P_1/2$, we have

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_0^{T_0} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}(T_0+1)^{1/2}k^{1/3}L^c,$$

$$\sum_{r\sim R} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}k^{1/3}T_1L^c,$$

$$T_0 < |T_1| \le T.$$

For the estimate of I_C it is enough to show that, for $k \le N^{3/20-\varepsilon}$ and $R \le P_2/2$, we have:

$$\sum_{\substack{r\sim R\\ k|r}} \sum_{\chi}{}^{*} \int_0^{T_0} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}(T_0+1)^{1/2}\left(\frac{R}{k}\right)^{1/6} L^c,$$

$$\sum_{\substack{r\sim R\\ k|r}} \sum_{\chi}{}^{*} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}\left(\frac{R}{k}\right)^{1/6} T_1L^c,$$

$$T_0 < |T_1| \le T.$$

These estimates follow from Lemma 5.1.

For the estimate of I_D it is enough to show that if $k \le N^{1/5-\varepsilon}$ and $R \le P_2/2k$, then

$$\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}(T_0+1)^{1/2}L^c,$$

$$\sum_{r\sim R}\sum_{\chi}{}^{*}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}T_1L^c,$$

$$T_0<|T_1|\le T.$$

These estimates follow from Lemma 5.1.

Likewise, for the proof of the estimate for I_E, we use Lemma 5.1 to show that for the estimate of I_C it is enough to show that for $k \le N^{3/20-\varepsilon}$ and $R \le P_2/2$, we have:

$$\sum_{\substack{r\sim R\\ k^2|r}}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}\left(\frac{R}{k^2}\right)^{1/6}(T_0+1)^{1/2}L^c,$$

$$\sum_{\substack{r\sim R\\ k^2|r}}\sum_{\chi}{}^{*}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll N^{1/2}\left(\frac{R}{k^2}\right)^{1/6}T_1L^c,$$

$$T_0<|T_1|\le T.$$

9. Proof of Theorem 3. Using (1.2) and Lemma 4.3, we derive an estimate for the number of the N-*exceptional zeros*. We find

$$\sum_{q\le N} N^*\left(1-\frac{EL_2}{L},q\right) \ll N^{((36/5)+\varepsilon)(EL_2/L)} \ll L^{36E/5+\varepsilon}.$$

Thus, there do not exist more than $\ll L^{36E/5+\varepsilon}$ N-*exceptional* integers. Each integer $\le N$ has at most $O(\log N)$ different prime factors. Thus, each N-*exceptional* integer does belong to at most $O(\log N)$ different sets A_k. Therefore, there are no more than $O(L^{36E/5+1+\varepsilon})$ prime numbers k, $1\le k\le N$, such that at least one of the integers $q\in A_k$ is N-exceptional.

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