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ON THE EXCEPTIONAL SET FOR THE SUM OF A PRIME AND THE k-TH POWER OF A PRIME

C. BAUER

1. Introduction

It is well known from the work of Montgomery and Vaughan that the exceptional set $E(x)$ for the binary Goldbach conjecture, i.e. the set of even numbers not larger than a real number x which are not representable as the sum of two primes, can be estimated by $E(x) \ll x^{1-\delta}$ for a $\delta > 0$. Brünner, Perelli, Pintz [1] and later Zaccagnini [14] applied the method of Montgomery and Vaughan to the problem of the representation of a positive integer as the sum of a prime and the k-th power of a natural number. They obtained an estimate for the corresponding exceptional set comparable to the one of Montgomery and Vaughan. In this paper we improve, for even integers satisfying certain congruence conditions, upon their result by giving the following theorem:

THEOREM. *Let*

$$E_k(x) = |n : n \leqq x,\ 2|n,\ n \not\equiv 1 (\operatorname{mod} p)\ \forall p > 2 \text{ with } p-1|k,\ n \neq p_1 + p_2^k\ \forall p_1, p_2 \in P|, \tag{1.1}$$

where P denotes the set of primes. Then there exists an effectively computable constant $\Theta = \Theta(k)$ such that

$$E_k(x) \ll_k x^{1-\Theta}.$$

After this article had been written, the author became aware that in a still unpublished work Liu and Shung [7] have also proved the above theorem. Even though both works are based on the circle method, we feel that our work

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is still of interest because our method differs essentially from the method used in [7]. We basically apply the method of [1] and [14] to our problem, whereas Liu and Shung use a method developed in [8]. Where we appeal to the lemmas 4.6–4.9 in order to calculate the contribution of the intervals over the major arcs, Liu and Shung apply a completely different technique of the Lemmas 3.1 to 3.4 in [8]. Furthermore, in their Lemma 4.6 they make use of Jordan's theorem on Dirichlet's integral which makes it necessary to extend the integration over the major arcs to infinity. Here, instead, we proceed differently by calculating precisely the effect of the P-*excluded* zeros (defined below).

2. Notation

To a certain extent we follow the notation and the structure of the proof in [14]. We define: $e(x)=e^{2\pi i x}$; x is a sufficiently large real number, p denotes a prime number, $s=\sigma+it$ is a complex number, $\varrho=\beta+i\gamma$ denotes the generic zeros of the L-functions. By $\chi(=\chi_q), \chi^*(=\chi_q^*), \chi_0(=\chi_{0,q})$ we denote a character, a primitive character and a principal character (modulo q), respectively, whereas $\chi \bmod q \longleftrightarrow \chi^* \bmod q^*$ indicates that the character χ is induced by the primitive character χ^* with $q^*|q$; cond χ =conductor of χ. We denote the Möbius function by $\mu(n)$, the Euler function by $\phi(n)$, the number of prime divisors of n by $\omega(n)$, the divisor function by $\tau(n)$, the cardinality of a set A by $|A|$ and the greatest common divisor and the smallest common multiple of the integers a and b by (a,b) and $[a,b]$, respectively. P is the set of prime numbers and for any integer $l \geqq 1$ we define

$$S_l(\alpha)=\sum_{\frac{\sqrt[l]{x}}{2}\leqq p<\sqrt[l]{x}} \log p\, e(\alpha p^l), \qquad S_l(\chi,\alpha)=\sum_{\frac{\sqrt[l]{x}}{2}\leqq p<\sqrt[l]{x}} \chi(p)\log p\, e(\alpha p^l),$$

$$T_\varrho(\alpha)=\sum_{\frac{x}{2}\leqq m<x} m^{\varrho-1}e(m\alpha), \qquad T(\alpha)=T_1(\alpha),$$

and for a fixed $k \geqq 2$ we define

$$F_\varrho(\alpha)=\sum_{\frac{\sqrt[k]{x}}{2}\leqq m<\sqrt[k]{x}} m^{\varrho-1}e(m^k\alpha), \qquad F(\alpha)=F_1(\alpha).$$

$$\sum_{\substack{\chi \bmod q\\ \chi \text{ primitiv}}} = \sum_{\chi \bmod q}{}^{*}, \qquad \sum_{\substack{a=1\\(a,q)=1}}^{q} = \sum_{a=1}^{q}{}^{*}, \qquad \sum_{a\leqq n\leqq b} u_n = \sum_{a}^{b} u_n,$$

$$C_l(\chi,a)=\sum_{m=1}^{q}{}^{*}\chi(m)e\left(\frac{m^l a}{q}\right), \qquad C_1(\chi,1)=\tau(\chi),$$

for a character χ modulo q.

$$A(q,n,\chi_1,\chi_2)=\sum_{a=1}^{q}{}^{*}C_1(\chi_1,a)C_k(\chi_2,a)e\left(\frac{-an}{q}\right),$$

for characters χ_1 and χ_2 modulo q.

$$A(q,n,\chi_{0,q},\chi_{0,q})=A(q,n), \qquad r(x,n)=\sum_{\substack{p_1+p_2^k=n\\ \frac{x}{2}\leqq p_1<x\\ \frac{\sqrt[k]{x}}{2}\leqq p_2<\sqrt[k]{x}}}\log p_1\log p_2,$$

$$L_{\varrho,\varrho'}(x,n)=\sum_{\substack{m+l^k=n\\ \frac{x}{2}\leqq m<x\\ \frac{\sqrt[k]{x}}{2}\leqq l<\sqrt[k]{x}}} m^{\varrho-1}l^{\varrho'-1}, \qquad L_{1,1}(x,n)=L(x,n),$$

$$\sigma(n,R,l)=\sum_{\substack{q\leqq R,\\ (q,l)=1}}\frac{A(q,n)}{\phi^2(q)}, \qquad \sigma(n,R)=\sigma(n,R,1),$$

$$N(\sigma,T,\chi)=|\{\sigma:\ L(\sigma,\chi)=0\ ,\beta\geqq\sigma,\ 0\leqq|\gamma|\leqq T\}|,$$

$$N^-(\sigma,P,T)=\sum_{q\leqq P}\sum_{\chi \bmod q}{}^{*}N(\sigma,T,\chi),$$

where the possibly existing Siegel zero (relative to P) is excluded.

$$N(n,q)(=N(q))=\Big|(m,l):m^k+l\equiv n(\bmod q), m,l\in\{1,2,\dots,q\},(ml,q)=1\Big|,$$

$$w(n,q)=\Big|m:m^k\equiv n(\bmod q), m\in\{1,2,\dots,q\},(m,q)=1\Big|.$$

c_1, c_2, $\dots$ as well as the O- and $\ll$- constants are effectively computable positive constants which may depend on k.

3. Preliminary results

In the following we only argue for a fixed number k. We first quote:

LEMMA 3.1. *There exists a positive constant $c_1<1$ such that $L(s,\chi)\neq 0$ in the region*

$$\sigma\geqq 1-\frac{c_1}{\log T}, \quad |t|\leqq T^{4k+7}$$

for all primitive characters $\chi \bmod q$, $q \leqq T$, $T \geqq 2$ *with the possible exception of at most one real primitive character* $\tilde{\chi} \bmod \tilde{r}$. *If it exists, the corresponding L-function has exactly one zero* $\tilde{\beta}$ *in the region given above, which is real, simple and satisfies*

$$\frac{c_2}{\tilde{r}^{1/2}\log^2\tilde{r}} \leqq 1-\tilde{\beta} \leqq \frac{c_1}{\log T}.$$

Furthermore, all the other zeros of the L-functions for primitive characters to moduls $q \leqq T$ *do not lie in the following region*

$$\sigma \geqq 1 - \frac{c_1}{\log T}\log\left(\frac{ec_1}{\delta(T)k(T)}\right), \ |t| \leqq T^{4k+7},$$

where $\delta(T)$ *and* $k(T)$ *are defined by*

$$\delta(T) = \begin{Bmatrix} (1-\tilde{\beta})\log T & \textit{if } \tilde{\beta} \textit{ exists,} \\ 1 & \textit{otherwise} \end{Bmatrix}, \quad k(T) = \begin{Bmatrix} 1 & \textit{if } \tilde{\beta} \textit{ exists,} \\ c_1 & \textit{otherwise} \end{Bmatrix}.$$

PROOF. [2], chapter 14 and [3], paragraph 4.

Set $P_1 = x^{b_1}$, where b_1 is a sufficiently small constant specified later. Let us further choose $T = P_1$ in Lemma 3.1. With the notation of Lemma 3.1 let further

$$P_2 = x^{b_2} = \begin{Bmatrix} P_1 \text{ if } \exists \tilde{r},\ \tilde{r} < P_1^{\lambda}, \\ P_1^{\lambda} \text{ otherwise} \end{Bmatrix},$$

where λ, $0 < \lambda = \lambda(k) < \frac{1}{2}$ is a sufficiently small parameter specified later. Then Lemma 3.1 holds with $T = P_2, \lambda c_1$ instead of c_1 and $\tilde{r} \leqq P_2^{\lambda}$ (if $\tilde{\beta}$ exists). We define the $P_2, \lambda c_1$-*-*excluded zeros* as those zeros $s = \sigma + it$ of the $L(s,\chi)$-functions, where χ is a primitive character $\bmod q$, $q \leqq P_2$, in the region

$$\sigma \geqq 1 - \frac{16k^2 \log\log x}{\log x}\log\left(e\left(\frac{2}{\delta(P_2)}\right)^{\frac{1}{\log\log x}}\right), \quad |t| \leqq P_2^{4k+7},$$

excluding the Siegel zero (relative to P_2) and $\delta(P_2)$ is defined by Lemma 3.1 with $T = P_2$ and λc_1 instead of c_1. (Here e does not denote the exponential function, but the number e.) For any number P with $P = P_2^{\eta}$ for an $\eta \in]0,1]$ holds Lemma 3.1, obviously with $T = P$ and $\eta\lambda c_1$ instead of c_1. The $P, \lambda\eta c_1$-*excluded zeros* are defined as the zeros of $L(s,\chi)$-functions to a primitive character $\chi \bmod q$, $q \leqq P$, in the region

$$\sigma \geqq 1 - \frac{16k^2 \log\log x}{\log x}\log\left(e\left(\frac{2(4k+2)}{(4k+3)\delta(P)}\right)^{\frac{1}{\log\log x}}\right), \quad |t| \leqq P^{4k+7},$$

excluding the Siegel zero (relative to P) and $\delta(P)$ is defined by Lemma 3.1 with $T = P$ and the constant $\eta\lambda c_1$. We estimate the number of $P, \lambda\eta c_1$-*excluded zeros* by means of

LEMMA 3.2. *There exist constants c_3 and c_4 such that*

$$N^-(\alpha, T, T^{4k+7}) \leqq c_3\delta(T)T^{c_4(1-\alpha)},$$

where $\delta(T)$ is defined as in Lemma 3.1.

PROOF. See Zaccagnini [36], Lemma 3.2.

Applying this lemma we get for a sufficiently small b:

$$N^-\left(1-\frac{16k^2\log\log x}{\log x}\log\left(e\left(\frac{2(4k+2)}{(4k+3)\delta(P)}\right)^{\frac{1}{\log\log x}}\right), P_2, P_2^{4k+7}\right)$$

$$\leqq c_3\delta(P_2)\exp\left(16k^2b_2c_4\log\log x - 16k^2b_2c_4\log\frac{\delta(P_2)(4k+3)}{2(4k+2)}\right)$$

$$\leqq \delta^{5/6}(P_2)\log^{1/6}x.$$

So we find by $\delta(P)\leqq 1$ that there are not more than

$$\ll \log^{1/3}x \tag{3.1}$$

pairs of numbers (ϱ, ϱ'), where each of the two numbers is an $P, \lambda\eta c_1$*-excluded zero* or a Siegel zero (relative to P) or $=1$. Now we prove that for every fixed P_2 we can find a P with $P = P_2^\eta$, $\eta \in \left[\frac{4k+2}{4k+3}, 1\right]$, for which further holds

$$\sigma \text{ is } P, \eta\lambda c_1 - \textit{excluded zero} \Rightarrow |\mathrm{Im}(\sigma)| \notin]P^{4k+3}, 16P^{4k+3}]. \tag{3.2}$$

First we have for a sufficiently large x and a fixed b_2:

$$16^{(\log x)^{1/6}} \leqq P_2^{1/4}. \tag{3.3}$$

Let $\{\gamma_1, \dots, \gamma_m\}$ be the imaginary parts of the $P_2, \lambda c_1 - * -$ *excluded zeros* with $|\gamma_i| \in [P_2^{4k+2}, P_2^{4k+3}]$ and $P_2^{4k+2} \leqq |\gamma_1| \leqq |\gamma_2| \leqq \dots \leqq |\gamma_m| \leqq P_2^{4k+3}$. Estimating the $P_2, \lambda c_1 - * -$ *excluded zeros* as in (3.1), we find by (3.3) that there holds at least one of the following three inequalities:

$$\exists t \in \{1, \dots, m-1\} \text{ with } \frac{|\gamma_{t+1}|}{|\gamma_t|} > 16 \quad \text{or} \quad \frac{P_2^{4k+3}}{|\gamma_m|} \geqq P_2^{1/4} \quad \text{or} \quad \frac{|\gamma_1|}{P_2^{4k+2}} \geqq P_2^{1/4}.$$

Setting in the first case $|\gamma_t| = P^{4k+3}$, in the second case $|\gamma_m| = P^{4k+3}$ and in the third case $P_2^{4k+2} = P^{4k+3}$, we find a P with $P \in [P_2^{4k+2/4k+3}, P_2]$. (If there holds more than one of the three inequalities, then the definition of P^{4k+3} can be chosen arbitrarily among the possible choices.) But by the definition of a $P, \eta\lambda c_1 -$ *excluded* and a $P_2, \lambda c_1 - * -$ *excluded* $-$ *zero* every

$P, \eta\lambda c_1$ – *excluded zero* is also an $P_2, \lambda c_1 - * -$ *excluded zero*, because by the definition of $\delta(P)$ and $\delta(P_2)$ by Lemma 3.1 with the constant $c_1\lambda\eta$ and $c_1\lambda$, respectively and by $\delta(P_2) \leqq 1$ (by Lemma 1) holds:

$$\frac{4k+2}{4k+3}\frac{1}{\delta(P)} \leqq \frac{1}{\delta(P_2)}.$$

So every $P, \eta\lambda c_1$ – *excluded zero*, which does not satisfy the condition (3.2), would be a $P_2, \lambda c_1 - * -$ *excluded zero*, which contradicts the choice of P. So P satisfies the condition (3.2). Then Lemma 3.1 holds with $T = P$, $c_1' = \eta\lambda c_1$ instead of c_1 and

$$\tilde{r} < P^{(4k+3/4k+2)\lambda} \tag{3.4}$$

(if the Siegel zero exists). In order to simplify the notation we write in the sequel $c_1' = c_1$ and the $P, \eta\lambda c_1$ – *excluded zeros* will be denoted as the P – *excluded zeros*. Let the P – *excluded characters* be the primitive characters $\chi(\bmod q), q \leqq P$, for which $L(s, \chi) = 0$, where s is a P – *excluded zero* and denote by the P – *excluded moduls* the moduls belonging to the P – *excluded characters*. We will also use the following notation:

$$\theta = \{P - excluded\ characters\}, \quad \theta' = \{P - excluded\ zeros\},$$

$$\begin{gathered} P = x^b,\ \delta(P) = \delta,\ \tilde{\chi} = \text{ exceptional character (to } P), \\ \tilde{\beta} = \text{ Siegel zero (to } P). \end{gathered} \tag{3.5}$$

The unit interval $\left[\frac{1}{Q}, 1 + \frac{1}{Q}\right]$ is now divided into the disjunct major arcs M and the minor arcs m, which are defined by

$$M = \sum_{q \leqq P} \sum_{a=1}^{q}{}^{*} I(a,q), \quad I(a,q) = \left[\frac{a}{q} - \frac{1}{Q}, \frac{a}{q} + \frac{1}{Q}\right],$$

$$m = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus M, \quad Q = xP^{-4k-3},$$

where P is defined by (3.2). We obtain

$$\begin{aligned} r(x,n) &= \int_{1/Q}^{1+(1/Q)} S(\alpha)S_k(\alpha)e(-n\alpha)d\alpha \\ &= \int_M S(\alpha)S_k(\alpha)e(-n\alpha) + \int_m S(\alpha)S_k(\alpha)e(-n\alpha) =: r_1(x,n) + r_2(x,n), \end{aligned} \tag{3.6}$$

where $r_1(x,n)$ and $r_2(x,n)$ are real, because the sets M and m are even mod 1.

4. Arithmetic and analytic lemmas

LEMMA 4.1. *Let $q = q_1 q_2$ and $(q_1, q_2) = 1$.*

(a) $N(q_1 q_2) = N(q_1) N(q_2)$.

(b) *For any prime number p and any natural number $\alpha \geqq 2$ holds: $N(p^\alpha) = p^{\alpha-1} N(p)$.*

(c) *For any natural number r holds:*

$$\frac{r}{\phi^2(r)} N(r) = \prod_{p|r} \frac{p}{(p-1)^2} N(p).$$

d) *Put $s(p,n) := 1 + \dfrac{A(p,n)}{(p-1)^2}$. Then we have*

$$s(p,n) = \frac{p}{(p-1)^2} N(p).$$

PROOF. (a) We note that every a with $1 \leqq a \leqq q$ can be written in a unique way as $a = a_1 q_2 + a_2 q_1$ with $1 \leqq a_i \leqq q_i$. We write

$$N(q) = \frac{1}{q} \sum_{a=1}^{q} \sum_{m=1}^{q}{}^{*} \sum_{l=1}^{q}{}^{*} e\left(\frac{m^k + l - n}{q} a\right),$$

split the summation over a in the two summations over a_1 and a_2 and after some arithmetical transformations get the lemma.

(b) By definition we have

$$N(p^\alpha) = \left| m : m^k \neq n (\operatorname{mod} p), m \in \{1, 2, \ldots, p^\alpha\}, (m,p) = 1 \right|.$$

For $\alpha \geqq 2$ we write for $(m,q) = 1$: $m = v + wp^{\alpha-1}$ with $1 \leqq v \leqq p^{\alpha-1}, (v,p) = 1$ and $0 \leqq w \leqq p-1$, from which we obtain

$$\begin{aligned} N(p^\alpha) &= \left| (v,w) : v^k \neq n (\operatorname{mod} p), 1 \leqq v \leqq p^{\alpha-1}, (v,p) = 1, 0 \leqq w \leqq p-1 \right| \\ &= pN(p^{\alpha-1}). \end{aligned}$$

Applying $(\alpha-2)$-times this argument we get part (b).

(c) We get from (a) and (b)

$$\begin{aligned} \frac{r}{\phi^2(r)} N(r) &= \prod_{p^\alpha \| r} \frac{p^\alpha}{\phi^2(p^\alpha)} N(p^\alpha) = \prod_{p|r} \frac{p^\alpha p^{\alpha-1}}{(p-1)^2 (p^{\alpha-1})^2} N(p) \\ &= \prod_{p|r} \frac{p}{(p-1)^2} N(p). \end{aligned}$$

(d)

$$s(p,n)=1+\frac{\sum_{a=1}^{p}{}^{*}\sum_{m=1}^{p}{}^{*}\sum_{l=1}^{p}{}^{*}e\left(\frac{m^k+l-n}{p}a\right)}{(p-1)^2}$$
$$=\frac{\sum_{a=1}^{p}\sum_{m=1}^{p}{}^{*}\sum_{l=1}^{p}{}^{*}e\left(\frac{m^k+l-n}{p}a\right)}{(p-1)^2}=\frac{p}{(p-1)^2}N(p).$$

LEMMA 4.2. *For any natural number $k\geqq 1$, any primitive character χ modulo p^α, $\alpha\geqq 1$ and $(a,p)>1$ holds:*

$$C_k(\chi,a)=0.$$

PROOF. Writing $a'=a/p$ and $m=u+vp^{\alpha-1}$ we obtain for $\alpha\geqq 2$:

$$C_k(\chi,a)=\sum_{m=1}^{p^\alpha}\chi(m)e\left(\frac{a'm^k}{p^{\alpha-1}}\right)=\sum_{u=1}^{p^{\alpha-1}}e\left(\frac{a'u^k}{p^{\alpha-1}}\right)\sum_{v=1}^{p}\chi(u+vp^{\alpha-1}),$$

which is equal to zero because the last inner sum vanishes for primitive characters. For $\alpha=1$ the lemma follows by the orthogonality relation of characters.

LEMMA 4.3. *For any natural number k, $q_1q_2=q$, $(q_2,q_1)=1$, $\chi_a(\bmod q)=\chi_{a_1}(\bmod q_1)\chi_{a_2}(\bmod q_2)$, $\chi_b(\bmod q)=\chi_{b_1}(\bmod q_1)\chi_{b_2}(\bmod q_2)$, and $h=h_1q_2+h_2q_1$*

(a) $C_k(\chi_a,h)=C_k(\chi_{a_1},h_1)C_k(\chi_{a_2},h_2)$.

(b) $A(q,n,\chi_a,\chi_b)=A(q_1,n,\chi_{a_1},\chi_{b_1})A(q_2,n,\chi_{a_2},\chi_{b_2})$.

(c) *For any natural number $k\geqq 1$, any primitive character χ modulo q, $q>1$ and $(a,q)>1$:*

$$C_k(\chi,a)=0.$$

PROOF. (a) is shown in the same way as Lemma 4.1 (a). Applying (a) we can show (b) in a similar way. (c) There exists a $p^\alpha\parallel q$, $\alpha\geqq 1$ with $(p,a)>1$. Writing $a=a_2p^\alpha+a_1\frac{q}{p^\alpha}$, it is by part (a) enough to prove that $C_k(\chi_{p^\alpha},a_1)=0$. But this follows from Lemma 4.2 because of $(p,a_1)>1$.

LEMMA 4.4. (a) *For any natural number n and prime number p*

$$A(p,n)=-(w(n,p)-1)p-1.$$

Let now be given any n which satisfies the congruence conditions in (1.1).

(b) *If at least one of the two characters* χ_1 *and* χ_2 *modulo* q, $q > 1$ *is primitive, then*

$$|A(q,n,\chi_1,\chi_2)| \leqq \phi^2(q) \prod_{p|q} \left(1 - \frac{(w(n,p)-1)p+1}{(p-1)^2}\right).$$

(c) *For any characters* χ_1 *and* χ_2 *modulo* q:

$$|A(q,n,\chi_1,\chi_2)| \ll \phi^2(q) \log^{4k} q.$$

(d) *For any prime number* p *and* $s(p,n)$ *defined as in Lemma 4.1* (d)

$$s(p,n) > 0$$

holds true.

PROOF. (a) By the definition of $A(p,n)$ we have

$$A(p,n) = -\sum_{a=1}^{p-1} \sum_{m=1}^{p-1} e\left(\frac{m^k - n}{p} a\right) = -(w(n,p)-1)\,p - 1.$$

(b) By Lemma 4.2 it holds:

$$\left|\phi^{-2}(q) A(q,n,\chi_1,\chi_2)\right| = \left|\phi^{-2}(q) \sum_{a=1}^{q} C_1(\chi_1,a) C_k(\chi_2,a) e\left(\frac{-an}{q}\right)\right|$$

$$= \left|\phi^{-2}(q) q \sum_{\substack{l+m^k \equiv n \,(\mathrm{mod}\, q),\ 1 \leqq l,m \leqq q, \\ (lm,q)=1}} \chi_1(l)\chi_2(m)\right|$$

$$\leqq \phi^{-2}(q) q N(q) \leqq \prod_{p|q} \phi^{-2}(p) p N(p),$$

where in the last step we have used Lemma 4.1 (c). Noting further that by the definition of $N(p)$ we have:

$$N(p) = \left|m:\ 1 \leqq m \leqq p-1,\ m^k \not\equiv n (\mathrm{mod}\, p)\right| = p - 1 - w(n,p), \tag{4.1}$$

we see that the lemma holds by

$$\left|\phi^{-2}(q) A(q,n,\chi_1,\chi_2)\right| \leqq \prod_{p|q} p \left(\frac{p-1-w(n,p)}{(p-1)^2}\right)$$

$$= \prod_{p|q} \left(1 - \frac{(w(n,p)-1)p+1}{(p-1)^2}\right).$$

(c) The lemma is trivial for $q=1$. If the characters χ_1 and χ_2 satisfy the condition of part (b), then part (c) follows from part (b), $w(n,p) \leqq k$ and so

$$\prod_{p|q} \left(1 - \frac{(w(n,p)-1)p+1}{(p-1)^2}\right) \leqq \prod_{p \leqq q} \left(1 + \frac{4k}{p}\right) \ll \log^{4k} q.$$

In the other case we have

$$\chi_1 = \chi_1^* \chi_{0,l} \quad \text{or} \quad \chi_1 = \chi_{0,q},$$

where $q = q^* l$ and χ_1^* is a primitive character modulo q^*, $q^* > 1$. We quote Lemma 5.3 in [9], which states that for a character χ modulo $q \leftrightarrow \chi^*$ modulo q^* and $(a,q)=1$ it holds

$$C_1(\chi, a) = \overline{\chi(a)} \tau(\chi^*) \mu\left(\frac{q}{q^*}\right) \chi^*\left(\frac{q}{q^*}\right). \tag{4.2}$$

So if $\chi_1 = \chi_1^* \chi_{0,l}$, we can restrict ourselves to the q which satisfies:

$$\mu(l) \neq 0, \quad (l, q^*) = 1. \tag{4.3}$$

From this we get $\chi_2 = \chi_3 \chi_4$ with $\chi_3 = \chi_3 \bmod q^*$ and $\chi_4 = \chi_4 \bmod l$. So we obtain from Lemma 4.3 (c) and the first part of the proof:

$$|A(q, n, \chi_1 \chi_2)| \ll \phi^2(q^*) \log^{4k} q^* A(l, n, \chi_{0,l}, \chi_4). \tag{4.4}$$

Using further the estimate

$$C_k(\chi, a) \ll_\epsilon q^{1/2+\epsilon}, \tag{4.5}$$

which holds for $(a,q)=1$ and may be found in [13], note to Lemma 4, we obtain

$$A(l, n, \chi_{0,l}, \chi_4) = \sum_{a=1}^{l}{}^{*} C_1(\chi_{0,l}, a) C_k(\chi_4, a) e\left(\frac{-an}{l}\right) = \ll_\epsilon l^{3/2+\epsilon}. \tag{4.6}$$

So the lemma follows from (4.4) and (4.6). If $\chi_1 = \chi_{0,q}$ the lemma follows immediately by arguing like in (4.6).

(d) By Lemma 4.1 (a) it is enough to show that $N(p) > 0$. Because of (4.1) the lemma is proved if $w(n,p) = 0$. In the other case we know (see Ireland, Rosen [5], p. 45) that $w(n,p) = (k, p-1)$, so that by (4.1) the lemma is proved in the case $p-1 \nmid k$. By Fermat's little theorem we know for $p-1|k$:

$$a^k \equiv 1 \pmod p \;\; \forall a \text{ with } a \not\equiv 0 \pmod p.$$

So we obtain for $p-1|k$:

$$n \equiv 1(\bmod p) \Longleftrightarrow w(n,p)=p-1 \Longleftrightarrow N(p)=0,$$

which proves the lemma.

LEMMA 4.5. *For two primitive characters* $\chi_1 \bmod q_1$ *and* $\chi_2 \bmod q_2$ *let* $q_3=[q_1,q_2]\leqq P$. *If* n *satisfies the congruence conditions in* (1.1), *there holds:*

$$\sum_{\substack{q\leqq P\\ q\equiv 0(\bmod q_3)}} \frac{|A(q,n,\chi_1\chi_{0,q},\chi_2\chi_{0,q})|}{\phi^2(q)} \ll \log^{5k+1} P.$$

PROOF. For $q_1|q$ let $q_1 l=q$. Analogously to (4.3) we only have to treat those q that satisfy

$$\mu(l)\neq 0, \quad (l,q_1)=1,$$

and for which, under the additional assumption $[q_1,q_2]=q_3$ and $q_3|q$,

$$\left(\frac{q}{q_3},q_3\right)=1 \tag{4.7}$$

holds. So we obtain

$$\chi_1\chi_{0,q}=\chi_1\chi_{0,\frac{q_3}{q_1}}\chi_{0,\frac{q}{q_3}}, \quad \chi_2\chi_{0,q}=\chi_2\chi_{0,\frac{q_3}{q_2}}\chi_{0,\frac{q}{q_3}},$$

and we further have by (4.2), Lemma 4.3 (b) and $w(p,n)\leqq k$:

$$|A(m,n)|\leqq \prod_{p|m} pk = mk^{\omega(m)}.$$

Using this, (4.7), Lemma 4.4 (c) and Lemma 4.5 in [14], we finally derive the lemma by

$$\sum_{\substack{q\leqq P\\ q\equiv 0(\bmod q_3)}} \frac{|A(q,n,\chi_1\chi_{0,q},\chi_2\chi_{0,q})|}{\phi^2(q)}$$
$$=\frac{\left|A(q_3,n,\chi_1\chi_{0,\frac{q_3}{q_1}},\chi_2\chi_{0,\frac{q_3}{q_2}})\right|}{\phi^2(q_3)} \sum_{\substack{m\leqq \frac{P}{q_3}\\ (m,q_3)=1}} \frac{|A(m,n)|}{\phi^2(m)}$$
$$\ll \log^{4k} P \sum_{m\leqq P} \frac{mk^{\omega(m)}}{\phi^2(m)} \ll \log^{4k+1} P \sum_{m\leqq P} \frac{k^{\omega(m)}}{m} \ll (\log P)^k.$$

LEMMA 4.6. *For all ϱ with $0 \leqq \mathrm{Re}(\varrho) \leqq 1$ and $s \geqq c_5 k^2 \log k$ it holds:*

$$\int_0^1 |F_\varrho(\alpha)|^{2s} d\alpha \ll x^{(2s/k)-1} .$$

PROOF. Considering the underlying diophantine equation this can be shown in the same way as Lemma 5.2 in [14].

LEMMA 4.7. (a) *Let $2^k x^{-1} < \lambda < x^{\frac{1}{k}-1}$ and $0 \leqq \mathrm{Re}(\varrho) \leqq 1$. There holds:*

$$\int_{-\lambda}^{\lambda} |F_\varrho(\alpha)|^2 \, d\alpha \ll x^{(2/k)-1}.$$

(b) *Let $2x^{-1} < \lambda < 1$ and $0 \leqq \mathrm{Re}\,(\varrho) \leqq 1$. There holds:*

$$\int_{-\lambda}^{\lambda} |T_\varrho(\alpha)|^2 \, d\alpha \ll x.$$

PROOF. (a) We define

$$u_n = \begin{cases} m^{\varrho-1} & \text{if } n = m^k \in [x/2^k, x[, \\ 0 & \text{otherwise.} \end{cases}$$

Then we get by Gallagher's lemma ([3], Lemma 1)

$$\int_{-\lambda}^{\lambda} |F_\varrho(\alpha)|^2 \, d\alpha \ll \int_{x/2^{k+1}}^{x} \left| \lambda \sum_{t}^{t+(2\lambda)^{-1}} u_n \right|^2 dt. \tag{4.8}$$

For the inner sum holds for a fixed $t \in [x/2^{k+1}, x]$

$$\sum_{t}^{t+(2\lambda)^{-1}} u_n \ll \sqrt[k]{(t+(2\lambda)^{-1}} - \sqrt[k]{t} \ll \lambda^{-1} x^{(1/k)-1}.$$

Substituting this in (4.8) we obtain the lemma. Part (b) is proved in the same way.

LEMMA 4.8. (a) *Let be given any $\sigma = \beta + i\gamma$ with $0 \leqq \beta \leqq 1$ and $|\gamma| \leqq x/Q$. Then for $1/Q \leqq |\alpha| \leqq 1/2$:*

$$T_\varrho(\alpha) \ll \frac{x^{\beta-1}}{|\alpha|}.$$

(b) *Let be given any* $\sigma = \beta + i\gamma$ *with* $0 \leqq \beta \leqq 1$ *and* $x^{1/2} \geqq |\gamma| > 16x/Q$. *Then for* $|\alpha| \leqq 1/Q$:

$$T_\varrho(\alpha) \ll \frac{x^\beta}{|\gamma|}.$$

PROOF. Part (a) is nearly identical to Lemma 12 in [1] and part (b) can be shown in the same way by appealing to Lemmas 4.2 and 4.8 in [11].

LEMMA 4.9. *If* $\varrho = \beta + i\gamma$ *with* $0 \leqq \beta \leqq 1$ *and* $|\gamma| \leqq P^{4k+3}$, *then for any* $s \geqq ck^2 \log k$ *and for all* ϱ' *with* $0 \leqq \mathrm{Re}(\varrho') \leqq 1$:

$$\int_{1/Q}^{1/2} |F_{\varrho'}(\alpha) T_\varrho(\alpha)| d\alpha \ll x^{(1/k)+\beta-1} P^{-\frac{2k+1}{s}}.$$

PROOF. Using Hölder's inequality, the Lemmata 4.6 and 4.8 (a) and the definition of Q this inequality can be shown in the same way as Lemma 5.8 in [14].

LEMMA 4.10. *If* $\varrho = \beta + i\gamma$ *with* $0 \leqq \beta \leqq 1$ *and* $16P^{4k+3} < |\gamma| \leqq P^{4k+7}$, *then there holds for all* ϱ' *with* $0 \leqq \mathrm{Re}(\varrho') \leqq 1$:

$$\int_{-\frac{1}{Q}}^{\frac{1}{Q}} |F_{\varrho'}(\alpha) T_\varrho(\alpha)| d\alpha \ll x^{1/k} P^{-2k-1}.$$

PROOF. Using the Lemmas 4.7 (a) and 4.8 (b) we get

$$\int_{-1/Q}^{1/Q} |F_{\varrho'}(\alpha) T_\varrho(\alpha)| d\alpha \ll \left(\int_{-1/Q}^{1/Q} |F_{\varrho'}(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_{-1/Q}^{1/Q} |T_\varrho(\alpha)|^2 d\alpha \right)^{1/2}$$
$$\ll x^{1/k} P^{-2k-1}.$$

5. Lemmas for the singular series

LEMMA 5.1. (a) *For any character* χ *modulo* p^{α_1} *and* $\alpha_1 \geqq 0$

$$C_k(\chi\chi_0, a) = 0$$

holds if χ_0 *is the principal character to the modulus* p^α, $p \nmid a$ *and* $\alpha \geqq j + \max(j, \alpha_1)$, *where* $j = 1 + \mathrm{ord}_k(p)$ *and* $w = \mathrm{ord}_k(p) \Longleftrightarrow p^w \parallel k$.

(b) *For any primitive character χ modulo p^α, $p \nmid a$, $w = \operatorname{ord}_k(p) \geqq 1$ and $\alpha \geqq 2w$ it holds:*

$$C_k(\chi, a) = 0.$$

(c) *Let χ be any primitive character modulo p^α for any prime number p and a natural number $\alpha \geqq 2$. Then there holds for any integer γ, $\alpha \geqq \gamma \geqq \alpha/2$:*

$$\chi\left(1+p^\gamma\right) = e\left(\frac{c}{p^{\alpha-\gamma}}\right),$$

where $c = c(\gamma)$, $1 \leqq c \leqq p^{\alpha-\gamma}$ is a natural number with $p \nmid c$.

(d) *Let χ be any primitive character modulo p^3 for any prime number $p < 2$. Then it holds*

$$\chi\left(1+p\right) = e\left(\frac{c}{p^2}\right),$$

where $1 \leqq c \leqq p^2$, $p \nmid c$.

Proof. (a) For $1 \leqq l \leqq p^\alpha$ we have $l = u + vp^{\alpha-j}$, $1 \leqq u \leqq p^{\alpha-j}$, $0 \leqq v \leqq p^j - 1$. By $\alpha \geqq j + \max(j, \alpha_1)$ is further $l^k \equiv u^k + vku^{k-1}p^{\alpha-j} \pmod{p^\alpha}$ and $l \equiv u \pmod{p^{\alpha_1}}$. So we get:

$$\begin{aligned} C_k(\chi\chi_0, a) &= \sum_{l=1}^{p^\alpha}{}^{*} \chi\chi_0(l) e\left(\frac{l^k a}{p^\alpha}\right) \\ &= \sum_{u=1}^{p^{\alpha-j}} \chi\chi_0(u) e\left(\frac{au^k}{p^\alpha}\right) \sum_{v=0}^{p^j-1} e\left(\frac{avku^{k-1}}{p^j}\right) = 0, \end{aligned}$$

because the inner sum vanishes for any p prime to u.

(b) We obtain in a similar way

$$C_k(\chi, a) = \sum_{l=1}^{p^\alpha}{}^{*} \chi(l) e\left(\frac{l^k a}{p^\alpha}\right) = \sum_{u=1}^{p^{\alpha-w}} e\left(\frac{au^k}{p^\alpha}\right) \sum_{v=0}^{p^w-1} \chi\left(u + vp^{\alpha-w}\right),$$

from which the lemma follows because the inner sum vanishes for a primitive character.

(c) It remains to show that $p \nmid c$. But if $p | c$, we obtain

$$\chi(1+ap^{\alpha-1}) = \chi^a(1+p^{\alpha-1}) = \chi^{ap^{\alpha-\gamma-1}}(1+p^\gamma) = e\left(\frac{acp^{\alpha-\gamma-1}}{p^{\alpha-\gamma}}\right) = 1,$$

which contradicts the primitivity of the character.

(d) Using $(1+p)^{p^2} \equiv 1 \left(\operatorname{mod} p^3\right)$ for $p \neq 2$ (see, e.g., Ireland, Rosen [5]; S. 43) and $p | \binom{p}{2}$ for $p \neq 2$ the proof is analogous to the one of part (c).

LEMMA 5.2. *In the parts* (a)–(d) *let be given a natural number* $q = p^\alpha$, $\alpha \geqq 1$, *two characters* χ_1 *and* $\chi_2 \bmod q$ *and* $p \nmid k$, $p^\alpha \nmid n$.

(a) *For* $q = p$, χ_1 *primitive and* $\chi_2 = \chi_{0,q}$ *it holds:*

$$A(p, n, \chi_1, \chi_2) \leqq (k+1)p^{3/2}.$$

(b) *For* $q = p$, χ_1 *primitive and* $\chi_2 \neq \chi_{0,q}$*:*

$$A(p, n, \chi_1, \chi_2) \leqq kp^{3/2}.$$

(c) *For* $q = p^\alpha$, $\alpha \geqq 4$, χ_1, χ_2 *primitive and* $p^\beta \parallel n$, $\beta \leqq \left[\frac{\alpha}{4}\right]$*:*

$$A(p^\alpha, n, \chi_1, \chi_2) \leqq kp^{\alpha + \left[\frac{\alpha+1}{2}\right] + \left[\frac{\alpha}{4}\right]}.$$

(d) *For* $q = p^\alpha$, $\alpha \in \{2, 3\}$, χ_1, χ_2 *primitive and under the additional conditions* $p \neq 2$ *and* $p^\beta \parallel n$, $\beta \leqq 1$ *in the case* $\alpha = 3$ *holds:*

$$A(p^2, n, \chi_1, \chi_2) \ll_\epsilon kp^{(7/4)\alpha + \epsilon}.$$

(e) *Let be given the principal character* $\chi_{0,\alpha}$ *to the module* p^α *and a primitive character* χ_2 *to the module* p^{α_1} *with* $\alpha_1 < \alpha$. *Let* $p^\beta \parallel n$, $\beta \leqq \left[\frac{\alpha_1}{4}\right]$. *If with the notation of Lemma 5.1* (a) $\alpha_1 \geqq \max(\alpha - \mathrm{ord}_k(p), 6, \frac{2}{3}\alpha)$, *then there holds for any primitive character* χ_1 *modulo* p^α*:*

$$A(p^\alpha, n, \chi_1, \chi_{0,\alpha}\chi_2) \leqq k^2 p^{\alpha + \left[\frac{\alpha+1}{2}\right] + \left[\frac{\alpha}{4}\right] + 1}.$$

PROOF. We first transform $A(q, n, \chi_1, \chi_2)$ (and $A(q, n, \chi_1, \chi_{0,\alpha}\chi_2)$). Noting that in Parts (a)–(e) χ_1 is always primitive, that $|\tau(\chi_1)| = q^{1/2}$ and that (4.2) also holds for $(a, q) > 1$ for primitive characters, we see

$$A(q, n, \chi_1, \chi_2) = \sum_{m,a=1}^{q}{}^{*} \chi_2(m) e\left(\frac{m^k - n}{q} a\right) \sum_{l=1}^{q}{}^{*} \chi_1(l) e\left(\frac{l}{q} a\right)$$

$$= \tau(\chi_1) \sum_{m=1}^{q}{}^{*} \chi_2(m) \sum_{a=1}^{q}{}^{*} e\left(\frac{m^k - n}{q} a\right) \overline{\chi_1(a)} \tag{5.1}$$

$$= |\tau(\chi_1)|^2 \sum_{m=1}^{q}{}^{*} \chi_1(m^k - n)\chi_2(m) = qD(\chi_1, \chi_2),$$

where $D(\chi_1, \chi_2) = \sum\limits_{m=1}^{q}{}^{*} \chi_1(m^k - n)\chi_2(m)$.

(a) This case follows immediately from (13.3) in [14] and (5.1).

(b) For any integer n which is prime to n we can write any character χ modulo p as

$$\chi(n) = e\left(\frac{m \operatorname{ind}_g(n)}{p-1}\right),$$

where $m \in \{1, \ldots, p-1\}$ and $\operatorname{ind}_g(n)$ denotes the index of n relative to a primitive root g of the reduced residue class system modulo p. Defining especially a character χ_s modulo p for $(n,p)=1$ by

$$\chi_s = e\left(\frac{\operatorname{ind}_g(n)}{p-1}\right)$$

(and $\chi_s(n) = 0$, if $(n,p) > 1$), we can write every character χ modulo p as $\chi = \chi_s^m$, $m \in \{1, \ldots, p-1\}$, where $m = p-1 \Longleftrightarrow \chi = \chi_0$. We obtain:

$$D(\chi_1, \chi_2) = \sum_{m=1}^{p} \chi_s^{m_1}(m^k - n)\chi_s^{m_2}(m) = \sum_{m=1}^{p} \chi_s\left(\left(m^k - n\right)^{m_1} m^{m_2}\right),$$

where m_1, $m_2 \in \{1, \ldots, p-2\}$. Let us denote F_p as the residue class system modulo p and $f(x) = (x^k - n)^{m_1} x^{m_2}$. With the notation of Theorem 2C′ in [10] (Weil's lemma) the character χ_s has the order $p-1$. If $f(x)$ is a $(p-1)$-th power in the sense of Theorem 2C′, every zero x_0 of $f(x) \in F_p[x]$ has the order $g_{x_0}(p-1)$, $g_{x_0} \in N$. Because of $p \nmid n$ and $m_2 \in \{1, \ldots, p-2\}$ the order of the zero $x_0 = 0$ is $\neq g_{x_0}(p-1)$. $f(x)$ not having more than $(k+1)$-different zeros, the lemma now follows from Theorem 2C′ in [10].

(c) Let $\gamma = \left[\frac{\alpha+1}{2}\right]$. Writing every number a with $1 \leqq a \leqq p^\alpha$ as $a = u + vp^\gamma$, $1 \leqq u \leqq p^\gamma, 0 \leqq v \leqq p^{\alpha-\gamma} - 1$ and noting that for every integer a, $p \nmid a$ there exists a number $\overline{a}$ with $a\overline{a} \equiv 1 (\operatorname{mod} p^\gamma)$, we get:

$$D(\chi_1, \chi_2) = \sum_{u=1}^{p^\gamma} \sum_{v=0}^{p^{\alpha-\gamma}-1} \chi_1(u^k - n + ku^{k-1}vp^\gamma)\chi_2(u + vp^\gamma)$$

$$= \sum_{u=1}^{p^\gamma} \chi_1(u^k - n)\chi_2(u) \sum_{v=0}^{p^{\alpha-\gamma}-1} \chi_1\left(1 + ku^{k-1}vp^\gamma\overline{(u^k - n)}\right)\chi_2(1 + \overline{u}vp^\gamma).$$

From this we obtain by Lemma 5.1 (c) and $(1+p^\gamma)^a \equiv 1 + ap^\gamma \,(\operatorname{mod} p^\alpha)$ for two natural numbers c_1 and c_2, which are defined by

$$\chi_i(1 + p^\gamma) = e\left(\frac{c_i}{p^{\alpha-\gamma}}\right), \qquad p \nmid c_i,\ i \in \{1,2\}: \tag{5.2}$$

(5.3)

$$D(\chi_1, \chi_2) = \sum_{u=1}^{p^\gamma} \chi_1\left(u^k - n\right)\chi_2(u) \sum_{v=0}^{p^{\alpha-\gamma}-1} e\left(\frac{c_1 ku^{k-1}v\overline{(u^k - n)}}{p^{\alpha-\gamma}}\right) e\left(\frac{c_2\overline{u}v}{p^{\alpha-\gamma}}\right).$$

From (5.2) and (5.3) it is obvious that $(c_1c_2k\overline{(u^k-n)}u,p)=1$. Noting further that $a\overline{a}\equiv 1(\bmod p^\gamma)\Longrightarrow a\overline{a}\equiv 1(\bmod p^{\alpha-\gamma})$, we see that the inner sum in (5.3) $\neq 0$ if

(5.4)

$$c_1ku^{k-1}\overline{(u^k-n)}+c_2\overline{u}\equiv 0(\bmod p^{\alpha-\gamma})\Longleftrightarrow u^k(c_1k+c_2)\equiv c_2n(\bmod p^{\alpha-\gamma}).$$

If $p^\beta\parallel n$ and $p^\delta\parallel c_1k+c_2$, there holds (by the assumption of the lemma) $\beta\leqq[\frac{\alpha}{4}]<\alpha-\gamma$. So because of $(uc_2,p)=1$ a necessary condition for the solvability of the last congruence is $\beta=\delta$, in which case we can equivalently examine the congruence

$$u^k\frac{c_1k+c_2}{p^\beta}\equiv c_2\frac{n}{p^\beta}(\bmod p^{\alpha-\gamma-\beta}),$$

which has mostly k solutions modulo $p^{\alpha-\gamma-\beta}$. So there are not more than $kp^{2\gamma-\alpha+\beta}$ numbers modulo p^γ for which the upper sum $\neq 0$. Together with (5.1) and (5.3) the lemma follows.

(d) We argue until (5.4) as in part (c). If p does not divide both n and c_1k+c_2, the congruence has not more than k solutions modulo $p^{\alpha-\gamma}$ and the result follows similarly to part (c). In the other case $p\parallel n$ and $p|c_1k+c_2$ we derive from (5.3) and (5.4):

$$D(\chi_1,\chi_2)=p\sum_{u=1}^{p^\gamma}\chi_1\left(u^k-n\right)\chi_2(u). \tag{5.5}$$

For any n prime to p we define

$$\chi_i(n)=e\left(\frac{m_i\operatorname{ind}_g(n)}{p^{\alpha-1}(p-1)}\right),$$

for $m_i\in\{1,\dots,p^{\alpha-1}(p-1)-1\}$ and $\operatorname{ind}_g(n)$ is the index of n relative to a primitive root g of the reduced residue system modulo p^α. Defining furthermore a character χ modulo p^α for $(n,p)=1$ by $\chi(n)=e\left(\frac{\operatorname{ind}_g(n)}{p^{\alpha-1}(p-1)}\right)$ (and $\chi(n)=0$, if $(n,p)>1$), we have

$$\chi_i=\chi^{m_i}. \tag{5.6}$$

χ is primitive by its definition, so we know by Lemma 5.1 (c) and (d) that $\chi(1+p)=e\left(\frac{c_3}{p^{\alpha-1}}\right)$ and $\chi_i(1+p)=e\left(\frac{c_i}{p^{\alpha-1}}\right)$, where $p\nmid c_i$, $i\in\{1,2,3\}$. By (5.6) it follows from this $c_i\equiv m_ic_3(\bmod p^{\alpha-1})$ $(i\in\{1,2\})$ and so:

$$p|c_1k+c_2\Longrightarrow p|m_1k+m_2. \tag{5.7}$$

By (5.5) and (5.6) we know furthermore

$$D(\chi_1,\chi_2) = p\sum_{u=1}^{p^\gamma} \chi^{m_1}\left(u^k - n\right)\chi^{m_2}(u) = p\sum_{u=1}^{p^\gamma}\chi^{m_1k+m_2}(u)\,\chi^{m_1}\left(1 - n\overline{u^k}\right),$$

where $\overline{u}$ is chosen such that $u^k\overline{u^k} \equiv 1(\text{mod } p^{\alpha-1} = \text{mod } p^\gamma)$, because so we get by $p \parallel n$: $nu^k\overline{u^k} \equiv n(\text{mod } p^\alpha)$. Furthermore, we know from (5.7) $p|m_1k+m_2$, from which we derive by $\gamma = \alpha - 1$ that

$$(h+p^\gamma)^{m_1k+m_2} \equiv h^{m_1k+m_2}(\text{mod } p^\alpha)\,\forall h \in N.$$

So we get

$$\chi^{m_1k+m_2}(h+p^\gamma) = \chi\left((h+p^\gamma)^{m_1k+m_2}\right) = \chi\left(h^{m_1k+m_2}\right) = \chi^{m_1k+m_2}(h),$$

which shows that $\chi^{m_1k+m_2}$ is a character modulo p^γ. For $\alpha = 2$ we get from the last identity for $D(\chi_1,\chi_2)$, $p \parallel n \iff n = \tilde{n}p$, $(\tilde{n}, p) = 1$, (5.6) and $\chi_1(1+p) = e\left(\frac{c_1}{p}\right)$:

$$D(\chi_1,\chi_2) = p\sum_{u=1}^{p}\overline{\chi}^{m_1k+m_2}(u)e\left(\frac{-\tilde{n}c_1u^k}{p}\right) \ll_\epsilon p^{3/2+\epsilon},$$

where the last inequality is derived by applying (4.5) to $\chi^{m_1k+m_2}$. If $\alpha = 2$ we can now derive the lemma by the last inequality and (5.1). If $\alpha = 3$ we write any $u \in \{1,\dots,p^2\}$ as $u = v + wp$, $1 \leqq v \leqq p$, $1 \leqq w \leqq p-1$, getting so by (5.6) and the second last identity derived for $D(\chi_1,\chi_2)$:

$$\begin{aligned} D(\chi_1,\chi_2) &= p\sum_{v=1}^{p}\sum_{w=0}^{p-1}\overline{\chi}^{m_1k+m_2}(v+wp)\chi_1\left(1 - \tilde{n}pv^k - \tilde{n}kv^{k-1}wp^2\right) \\ &= p\sum_{v=1}^{p}\overline{\chi}^{m_1k+m_2}(v)\,\chi_1\left(1-\tilde{n}pv^k\right)\sum_{w=0}^{p-1}\overline{\chi}^{m_1k+m_2} \\ &\times(1+\overline{v}wp)\,\chi_1\left(1 - \left(\overline{1-\tilde{n}pv^k}\right)\tilde{n}kv^{k-1}wp^2\right), \end{aligned}$$

where $a\overline{a} \equiv 1(\text{mod } p)$, which implies $v\overline{v}wp \equiv wp(\text{mod } p^2)$, which is sufficient, because $\overline{\chi}^{m_1k+m_2}$ has been shown to be a character modulo $p^{\alpha-1} = p^2$, and implies also $\left(1-\tilde{n}pv^k\right)\left(\overline{1-\tilde{n}pv^k}\right)\tilde{n}kv^{k-1}wp^2 \equiv \tilde{n}kv^{k-1}wp^2(\text{mod } p^3)$. We know by Lemma 5.1 (c) that

$$\chi_1(1+p^2) = e\left(\frac{c_4}{p}\right), \quad \overline{\chi}(1+p^2) = e\left(\frac{c_5}{p}\right), \quad p \nmid c_4c_5,$$

and, in general,

$$\chi_a\left(1+bp^2\right)=\chi_a^b\left(1+p^2\right),\quad \chi_a\in\{\chi_1,\overline{\chi}\}.$$

From (5.7) we know further that $m_1k+m_2=pc_6$ and so we get by $p|\binom{m_1k+m_2}{2}$ for $p>2$

$$(1+\overline{v}wp)^{m_1k+m_2}\equiv 1+\overline{v}wc_6p^2(\text{mod } p^3),$$

from which we derive together with the last identity for $D(\chi_1,\chi_2)$:

$$D(\chi_1,\chi_2)=p\sum_{v=1}^{p}\overline{\chi}^{m_1k+m_2}(v)\,\chi_1\left(1-\tilde{n}pv^k\right)\\ \times\sum_{w=0}^{p-1}e\left(w\frac{c_5c_6\overline{v}-c_4\left(\overline{1-\tilde{n}pv^k}\right)\tilde{n}kv^{k-1}}{p}\right).$$

Similarly to part (c) we concentrate on the congruence

$$c_5c_6\overline{v}-c_4\left(\overline{1-\tilde{n}pv^k}\right)\tilde{n}kv^{k-1}\equiv 0(\text{mod } p),$$

which for $p|c_6$ is not solvable because of $\left(c_4\left(\overline{1-\tilde{n}pv^k}\right)\tilde{n}kv^{k-1},p\right)=1$ and in the other case is equivalent to

$$\Longleftrightarrow v^k\left(-c_5c_6\tilde{n}p-c_4\tilde{n}k\right)+c_5c_6\equiv 0(\text{mod } p).$$

By $(c_4c_5c_6\tilde{n}k,p)=1$ this congruence has at most k solutions modulo p, from which the lemma follows together with (5.1) for $\alpha=3$.

(e) Define $\lambda=\left[\frac{\alpha_1+1}{2}\right]+1$. We write a with $1\leqq a\leqq p^\alpha$ as $a=u+vp^\lambda$, $1\leqq u\leqq p^\lambda$, $0\leqq v\leqq p^{\alpha-\lambda}-1$. By the assumptions of the lemma we have $k=\tilde{k}p^{\alpha-\alpha_1+d}$, with $(\tilde{k},p)=1$, $d\geqq 0$ and for $b\geqq 3$

$$p^{2\lambda}\binom{k}{2}\equiv p^{b\lambda}\equiv 0\,(\text{mod } p^\alpha).$$

Using this we get as in part (c)

$$D(\chi_1,\chi_{0,\alpha}\chi_2)=\sum_{u=1}^{p^\lambda}\chi_1\left(u^k-n\right)\chi_{0,\alpha}\chi_2(u)\\ \times\sum_{v=0}^{p^{\alpha-\lambda}-1}\chi_1\left(1+\tilde{k}p^{\alpha-\alpha_1+d}u^{k-1}vp^\lambda\overline{(u^k-n)}\right)\chi_{0,\alpha}\chi_2\left(1+\overline{u}vp^\lambda\right),$$

where $\overline{a}$ is chosen such that $a\overline{a} \equiv 1 (\bmod p^\lambda)$, in which way we get:

$$(u^k - n)\overline{(u^k - n)}\tilde{k}p^{\alpha-\alpha_1+d}u^{k-1}vp^\lambda \equiv \tilde{k}p^{\alpha-\alpha_1+d}u^{k-1}vp^\lambda (\bmod p^\alpha)$$

and $\overline{u}uvp^\lambda \equiv vp^\lambda (\bmod p^{\alpha_1})$. By $\alpha - \alpha_1 + \lambda \geqq \frac{\alpha}{2}$ we get by Lemma 5.1 (c) and $\chi_{0,\alpha}\chi_2(m) = \chi_2(m)\,\forall m$ analogously to (5.2)

$$\chi_1(1 + p^{\alpha-\alpha_1+\lambda}) = e\left(\frac{c_1}{p^{\alpha_1-\lambda}}\right),$$
$$\chi_{0,\alpha}\chi_2(1 + p^\lambda) = \chi_2(1 + p^\lambda) = e\left(\frac{c_2}{p^{\alpha_1-\lambda}}\right),$$

where $p \nmid c_1c_2$. We obtain as in (5.3)

$$D(\chi_1, \chi_{0,\alpha}\chi_2) = \sum_{u=1}^{p^\lambda} \chi_1\left(u^k - n\right)\chi_{0,\alpha}\chi_2(u)$$
$$\times \sum_{v=0}^{p^{\alpha-\lambda}-1} e\left(\frac{c_1\tilde{k}p^d u^{k-1}v\overline{(u^k - n)}}{p^{\alpha_1-\lambda}}\right) e\left(\frac{c_2\overline{u}v}{p^{\alpha_1-\lambda}}\right).$$

Arguing as before we see that because of $(c_2u, p) = 1$ the inner sum can only be $\neq 0$ if $d = 0$, in which case we have to examine the congruence

$$u^k(c_1\tilde{k} + c_2) \equiv c_2n (\bmod p^{\alpha_1-\lambda}).$$

By $\beta < \alpha_1 - \lambda$ it is equivalent to the congruence

$$\frac{u^k(c_1\tilde{k} + c_2)}{p^\beta} \equiv \frac{c_2n}{p^\beta} (\bmod p^{\alpha_1-\lambda-\beta}),$$

that has at most k solutions modulo $p^{\alpha_1-\lambda-\beta}$, from which the lemma follows similarly to part (c).

LEMMA 5.3. *For any two primitive characters $\chi_1 \bmod q_1$ and $\chi_2 \bmod q_2$ with $q_3 = [q_1, q_2] \leqq x^{\frac{1}{4}}$ holds for all but $\ll xq_3^{-1/16}$ natural numbers $n \in [(9/10)x, x[$:*

$$A(q_3, n, \chi_1\chi_{0,q_3}, \chi_2\chi_{0,q_3}) \ll q_3^{2-(1/32)}.$$

PROOF. The case $q_3 = 1$ is trivial. As in (4.3) we can concentrate on the case

$$q_3 = q_1q_4,\ (q_1, q_4) = 1,\ \chi_1\chi_{0,q_3} = \chi_1\chi_{0,q_4},$$
$$\chi_2\chi_{0,q_3} = \chi_5\chi_6 \text{ with } \chi_5 \bmod q_1,\ \chi_6 \bmod q_4.$$

By applying Lemma 4.3 (b) and arguing as in (4.6) we obtain

$$\begin{aligned} &A(q_3, n, \chi_1\chi_{0,q_3}, \chi_2\chi_{0,q_3}) = A(q_1, n, \chi_1, \chi_5)A(q_4, n, \chi_{0,q_4}, \chi_6), \\ &A(q_4, n, \chi_{0,q_4}, \chi_6) \ll q_4^{(3/2)+\epsilon}. \end{aligned} \tag{5.8}$$

The lemma follows from (5.8), if χ_1 is the principal character to a module $q_1 \leqq q_3^{3/4}$, because in this case we get by (5.1) and (5.8):

$$\begin{aligned} |A(q_3, n, \chi_1\chi_{0,q_3}, \chi_2\chi_{0,q_3})| &\ll q_1^2 \left(\frac{q_3}{q_1}\right)^{(3/2)+\epsilon} \\ &\leqq q_1^{1/2} q_3^{(3/2)+\epsilon} \leqq q_3^{(15/8)+\epsilon} \leqq q_3^{2-(1/32)}. \end{aligned}$$

So we assume in the following that χ_1 is a primitive character to a module $q_1 > q_3^{3/4}$. By Lemma 4.3 (b) we have

(5.9)

$$A(q_1, n, \chi_1, \chi_5) = \prod_{D\in\{A,B,C\}} \prod_{i=1}^{3} \prod_{\substack{p^\alpha \| q_1 \\ A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in D_i}} A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}),$$

where $\chi_i = \prod_{p^\alpha \| q_1} \chi_{i,p^\alpha}$, $i \in \{1, 5\}$, χ_{1,p^α} mod p^α, an empty product is equal to 1 and

$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in A_1 \Longleftrightarrow \alpha = 1,\ p|k,$
$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in A_2 \Longleftrightarrow \alpha = 1,\ p\nmid kn,$
$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in A_3 \Longleftrightarrow \alpha = 1,\ p\nmid k,\ p|n,$
$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in B_1 \Longleftrightarrow \chi_{5,p_\alpha}$ primitive, $\alpha \geqq 2$, $p|k$,
$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in B_2 \Longleftrightarrow \chi_{5,p_\alpha}$ primitive, $\alpha = 2$, $p\nmid k$, $p^\alpha \nmid n$ or χ_{5,p_α} primitive, $\alpha = 3$, $p\nmid k$, $p \neq 2$, $p^\beta \| n$ with $\beta \leqq 1$ or χ_{5,p_α} primitive, $\alpha \geqq 4$, $p\nmid k$, $p^\beta \| n$ with $\beta \leqq \left[\frac{\alpha}{4}\right]$,
$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in B_3 \Longleftrightarrow \chi_{5,p_\alpha}$ primitive, $\alpha \geqq 2$, $A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \notin B_1 \cup B_2$,
$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in C_1 \Longleftrightarrow \chi_{5,p_\alpha}$ not primitive, $\alpha \geqq 2$, $p^\beta \| n$ with $\beta > \left[\frac{\alpha}{6}\right]$,
$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in C_2 \Longleftrightarrow \chi_{5,p_\alpha}$ not primitive, $\alpha \geqq 2$, $p^\beta \| n$ with $\beta \leqq \left[\frac{\alpha}{6}\right]$, cond $\chi_{5,p_\alpha} \geqq \max\left(\mathrm{ord}_k(p)+1, 6, \frac{2}{3}\alpha\right)$,

$$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in C_3 \Longleftrightarrow \chi_{5,p_\alpha} \text{ not primitive, } \alpha \geqq 2, \quad A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \notin C_1 \cup C_2.$$

For $A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in A_3 \cup B_3 \cup C_1$ we have by (5.1) trivially:

$$|A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha})| \leqq p^{2\alpha}. \tag{5.10}$$

In the following let cond $\chi_{5,p^\alpha} = \alpha_1$. For the estimation of $A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in C_2$, by Lemma 5.1 (a) and by the relation $\operatorname{ord}_k(p) + 1 \leqq \alpha_1$, which holds by the definition of C_2, we can restrict our observations to the case $\alpha \leqq \operatorname{ord}_k(p) + \alpha_1$. By $\beta \leqq \left[\frac{\alpha}{6}\right] \leqq \left[\frac{\alpha_1}{4}\right]$ the conditions of Lemma 5.2 (e) are satisfied in this case. So we get by Lemma 5.2 (a)–(e) for $A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in A_2 \cup B_2 \cup C_2$:

$$A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \leqq c_6 k^2 p^{(17/9)\alpha}. \tag{5.11}$$

For the estimation $A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in C_3$, by Lemma 5.1 (a), we have only to look at the case $\alpha \leqq \operatorname{ord}_k(p) + \max(\operatorname{ord}_k(p) + 1, \alpha_1)$ and so $\operatorname{ord}_k(p) \geqq 1$. If the maximum on the right side is $\operatorname{ord}_k(p) + 1$, we have

$$\alpha \leqq 3 \operatorname{ord}_k(p).$$

In the other case it follows from the definition of C_3

$$\alpha_1 < \max\left(\operatorname{ord}_k(p) + 1, 6, \frac{2}{3}\alpha\right) \leqq \max\left(6 \operatorname{ord}_k(p), \frac{2}{3}\left(\operatorname{ord}_k(p) + \alpha_1\right)\right),$$

from which together with the equivalence

$$\alpha_1 < \frac{2}{3}\left(\operatorname{ord}_k(p) + \alpha_1\right) \Longleftrightarrow \alpha_1 < 2 \operatorname{ord}_k(p)$$

it follows that:

$$\alpha_1 < 6 \operatorname{ord}_k(p) \text{ and so } \alpha \leqq 6 \operatorname{ord}_k(p).$$

So we get in both cases

$$\alpha \leqq 6 \operatorname{ord}_k(p),$$

from which we get together with Lemma 5.1 (b) for $A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in A_1 \cup B_1 \cup C_3$:

$$|A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha})| \leqq p^{18 \operatorname{ord}_k(p)}. \tag{5.12}$$

We define now

$$f(q_1, n) = \prod_{\substack{p^\alpha \| q_1 \\ A(p^\alpha, n, \chi_{1,p^\alpha}, \chi_{5,p^\alpha}) \in A_1 \cup B_1 \cup C_3}} p^\alpha,$$

$$g(q_1,n)=\prod_{\substack{p^\alpha \| q_1\\ A(p^\alpha,n,\chi_{1,p^\alpha},\chi_{5,p^\alpha})\in A_3\cup B_3\cup C_1}} p^\alpha,$$

$$h(q_1,n)=\prod_{\substack{p^\alpha \| q_1\\ A(p^\alpha,n,\chi_{1,p^\alpha},\chi_{5,p^\alpha})\in A_2\cup B_2\cup C_2}} p^\alpha.$$

Then we have $f(q_1,n)g(q_1,n)h(q_1,n)=q_1$, $g(q_1,n)\leqq 8(q_1,n)^6$ and the three factors are pairwise prime. Defining characters $\chi_{q_i,d} \bmod d(q_1,n)$, $i\in\{1,5\}$, $d\in\{f,g,h\}$ with $\chi_i=\prod_{d\in\{f,g,h\}}\chi_{i,d}$, we get by Lemma 4.3 (b) and (5.9)–(5.12):

$$\begin{aligned}
|A(q_1,n,\chi_1,\chi_5)| &= \prod_{d\in\{f,g,h\}} |A(d(q_1,n),n,\chi_{1,d},\chi_{2,d})|\\
&\leqq k^{18}(c_6k^2)^{\omega(q_1)}g^2(q_1,n)h^{17/9}(q_1,n)\\
&\ll (c_6k^2)^{\omega(q_1)}g^{1/9}(q_1,n)q_1^{17/9}\\
&\ll (c_6k^2)^{\omega(q_1)}(q_1,n)^{2/3}q_1^{17/9}.
\end{aligned}\tag{5.13}$$

Let

$$\begin{aligned}
A(x,q_1) &= \left|n\in[(9/10)x,x[,\ (q_1,n)\geqq q_1^{1/10}\right|,\\
B(q_1) &= \left|m \bmod q_1,\ (q_1,m)\geqq q_1^{1/10}\right|.
\end{aligned}$$

Then we have obviously

$$A(x,q_1)\ll\left(\frac{x}{q_1}+1\right)B(q_1),$$

and

$$B(q_1)\leqq\sum_{\substack{d\mid q_1\\ d\geqq q_1^{1/10}}}\frac{q_1}{d}\leqq\tau(q_1)q_1^{9/10},$$

from which we deduce

$$A(x,q_1)\ll xq_1^{-1/10}\tau(q_1)\leqq xq_3^{-3/40}\tau(q_3)\ll xq_3^{-1/16}.\tag{5.14}$$

The lemma follows now from (5.8), (5.13), (5.14) and $(c_6k^2)^{\omega(q_1)}\ll_\epsilon q_1^\epsilon$.

LEMMA 5.4. *For all n and all l holds:*

$$\sigma(n,R,l)\ll(\log R)^{k+1}.$$

PROOF. From Lemmas 4.3 (b), 4.4 (a), (4.2) and Lemma 4.5 in [14] it follows that

$$|\sigma(n,R,l)| \leqq \sum_{\substack{q\leqq R\\(q,l)=1}} \frac{\mu^2(q)|A(q,n)|}{\phi^2(q)} \leqq \sum_{q\leqq R}\frac{qk^{\omega(q)}}{\phi^2(q)} \ll \log R \sum_{q\leqq R}\frac{k^{\omega(q)}}{q} \ll (\log R)^{k+1}.$$

LEMMA 5.5. *Let $P=x^d$, where d is a positive constant $\leqq 1/10$. Let be given a set of natural numbers l_i, $1\leqq i\leqq s\ll(\log x)^{1/3}$, with $\frac{P}{l_i}\geqq P^{\frac{4}{5}}$. Then for sufficiently small d there holds*

$$\sigma\left(n,\frac{P}{l_i},l_i\right)=\prod_{\substack{p\leqq P\\(p,l_i)=1}}\left(1+\frac{A(p,n)}{(p-1)^2}\right)+O\left(P^{-\frac{1}{16}}\right)$$

for all but $\ll x^{1-\delta_1}$, $\delta_1\geqq 0$ natural numbers $n\in[(9/10)x,x[$, which satisfy the congruence conditions in (1.1), and for all $i\in\{1,\dots,s\}$.

PROOF. The congruence conditions for n are required because of Lemma 4.5 (c). We first argue for a fixed $l\in\{l_1,\dots,l_s\}$ and set $\frac{P}{l}=R$. Defining $A(q,n,l)=\mu((q,l)^2)A(q,n)$ and noting Lemma 4.3 (b) and (4.2), we obviously have to estimate:

$$\begin{aligned}&\left|\sum_{q\leqq R}\frac{A(q,n)}{\phi^2(q)}-\prod_{p\leqq P}\left(1+\frac{A(p,n)}{(p-1)^2}\right)\right|\\ &\leqq\left|\sum_{\substack{R<q<V\\q\in\mathcal{D}}}\phi^{-2}(q)A(q,n,l)\right|+\left|\sum_{\substack{q\geqq V\\q\in\mathcal{D}}}\phi^{-2}(q)A(q,n)\right|\\ &=: T_1(n,R)+T_2(n,R),\end{aligned}\tag{5.15}$$

where $V=\exp\left(\frac{\log P\log x}{\log\log x}\right)$ and

$$\mathcal{D}=\{q:\, q\in N,\ \mu(q)\neq 0,\ p|q\Rightarrow p\leqq P\}.$$

We first estimate $T_1(n,R)$. We have:

$$\phi^{-2}(q)A(q,n)=\phi^{-2}(q)\sum_{m|q}A_1(m,n)A_2(q/m,n),\tag{5.16}$$

where we define by Lemma 4.4 (a) and $w(n,p)=0$ for $p|n$:

$$A_1(p,n)=\begin{cases}-\mu((p,l)^2)p(w(p,n)-1) & p\nmid n,\\ 0 & p|n,\end{cases}$$

$$A_2(p,n,l) = \begin{cases} -\mu((p,l)^2) & p\nmid n, \\ \mu((p,l)^2)(p-1) & p|n, \end{cases}$$
$$A_i(q,n) = \prod_{p|q} A_i(p,n), i \in \{1,2\},$$

and an empty product is equal to 1. For $p \nmid n$ it holds

$$w(n,p) = \Big| m : m^k \equiv n (\bmod p), m \in (1,2,\ldots,p) \Big|.$$

We obtain by Lemma 4.3 in [13], (4.2) and $|\tau(\chi)| \leqq p^{1/2}$ for $p \nmid n$:

$$\begin{aligned} w(n,p) &= 1 + \frac{1}{p}\sum_{a=1}^{p-1} e\left(\frac{-n}{p}a\right)\sum_{m=1}^{p} e\left(\frac{m^k}{p}a\right) \\ &= 1 + \frac{1}{p}\sum_{\chi\in\mathcal{A}(p)} \tau(\chi)\sum_{a=1}^{p-1} e\left(\frac{-n}{p}a\right)\overline{\chi(a)} \\ &= 1 + \frac{1}{p}\sum_{\chi\in\mathcal{A}(p)} |\tau(\chi)|^2\chi(-n) = 1 + \sum_{\chi\in\mathcal{A}(p)} \chi(-n), \end{aligned}$$

where $\mathcal{A}(p)$ denotes the set of non-principal characters χ modulo p, for which χ^k is the principal character and

$$|\mathcal{A}(p)| = (k, p-1) - 1. \tag{5.17}$$

So we deduce for all p:

$$A_1(p,n) = -\mu((p,l)^2)p\sum_{\chi\in\mathcal{A}(p)} \chi(-n). \tag{5.18}$$

We obtain from (5.15) and (5.16)

$$\begin{aligned} T_1(n,R) \leqq& \sum_{\substack{R^{1/3}<m<V \\ m\in\mathcal{D}}} \phi^{-2}(m)|A_2(m,n)| \sum_{\substack{R/m<d<V/m,\,(d,m)=1 \\ d\in\mathcal{D}}} \phi^{-2}(d)\,|A_1(d,n)| \\ &+ \sum_{\substack{m\leq R^{1/3} \\ m\in\mathcal{D}}} \phi^{-2}(m)|A_2(m,n)| \left|\sum_{\substack{R/m<d<V/m\ (d,m)=1 \\ d\in\mathcal{D}}} \phi^{-2}(d)A_1(d,n)\right| \\ =:& F_1(n,R) + F_2(n,R). \end{aligned} \tag{5.19}$$

For $F_1(n,R)$ we get by $w(n,p) \leqq k$:

$$F_1(n,R) \leqq R^{-1/3} \sum_{\substack{m<V \\ m\in\mathcal{D}}} \phi^{-2}(m) m |A_2(m,n)| \sum_{\substack{d<V \\ d\in\mathcal{D}}} \phi^{-2}(d)|A_1(d,n)|$$

$$\leqq R^{-1/3} \prod_{\substack{p\leqq P \\ p\nmid n}} \left(1+\frac{p|A_2(p,n)|}{(p-1)^2}\right) R^{-1/3} \prod_{\substack{p\leqq P \\ p\mid n}} \left(1+\frac{p|A_2(p,n)|}{(p-1)^2}\right)$$

$$\times \prod_{p\leqq P}\left(1+\frac{|A_1(p,n)|}{(p-1)^2}\right) \tag{5.20}$$

$$\leqq R^{-1/3} \prod_{p\leqq P}\left(1+\frac{4}{p}\right) 3^{\omega(n)} \prod_{p\leqq P}\left(1+\frac{4(k-1)}{p}\right)$$

$$\ll R^{-1/3}(\log P)^{4k} 3^{\omega(n)}.$$

For the estimation of $F_2(n,R)$ we obtain by the definition of $A_1(d,n)$ and (5.18):

$$\sum_{\substack{R/m<d<V/m,\,(d,m)=1 \\ d\in\mathcal{D}}} \phi^{-2}(d) A_1(d,n)$$

$$= \sum_{\substack{R/m<d<V/m,\,(d,m)=1 \\ d\in\mathcal{D}}} \prod_{p\mid d}\left(-\frac{\mu((p,l)^2)}{(p-1)^2} p \sum_{\chi\in\mathcal{A}(p)} \chi(-n)\right) \tag{5.21}$$

$$= \sum_{\substack{R/m<d<V/m \\ d\in\mathcal{D}}} \sum_{\chi \bmod d}{}^{*} f(\chi)\chi(-n),$$

where

$$f(\chi) = \begin{cases} \prod_{p\mid d}\left(-\frac{p}{(p-1)^2}\right) & \text{if } \chi = \prod_{p\mid d}\chi_p \text{ with } \chi_p\in\mathcal{A}(p)\ \forall p\mid d,\ (ml,d)=1, \\ 0 & \text{otherwise.}\end{cases}$$

By (5.17) we find for any positive number a and any $d\in\mathcal{D}$:

$$\sum_{\chi \bmod d}{}^{*} |f(\chi)|^a \leqq (k-1)^{\omega(d)} \left(\prod_{p\mid d}\frac{p}{(p-1)^2}\right)^a \leqq \frac{(4^a(k-1))^{\omega(d)}}{d^a}. \tag{5.22}$$

Now we get from (5.21):
(5.23)

$$\sum_{\substack{R/m<d<V/m,\,(d,m)=1 \\ d\in\mathcal{D}}} \phi^{-2}(d) A_1(d,n,l) = \sum_{j=1}^{L} \sum_{\substack{Q_{j-1}<d\leqq Q_j, \\ d\in\mathcal{D}}} \sum_{\chi \bmod d}{}^{*} f(\chi)\chi(-n),$$

where $Q_0 = R/m$, $Q_j = x^{j/2}$, $j = 1, \dots, L$, $L \leqq 2\dfrac{\log P}{\log\log x}$.

We have for a fixed j by (5.22), Lemma 6.5 in [6] and Lemma 4.5 in [14]:

$$\sum_{n\in[(9/10)x,x[} \left| \sum_{\substack{Q_{j-1}<d\leqq Q_j \\ d\in\mathcal{D}}} \sum_{\chi \bmod d}{}^{*} f(\chi)\chi(-n) \right|$$

$$\ll \left(x^{1/2} + Q_j^{1/j}\right) x^{1/2} \left(\log\left(x^j e\right)\right)^{(j^2-1)/2j}$$

(5.24)

$$\times \left(\sum_{Q_{j-1}<d\leqq Q_j,\, \chi \bmod d} \sum{}^{*} |f(\chi)|^{2j/(2j-1)} \right)^{(2j-1)/2j}$$

$$\ll x\left(\log\left(x^j e\right)\right)^{(j^2-1)/2j} \left(\frac{1}{Q_{j-1}^{1/(2j-1)}} \sum_{Q_{j-1}<d\leqq Q_j} \frac{(16k)^{\omega(d)}}{d} \right)^{(2j-1)/2j}$$

$$\ll \left(\frac{1}{Q_{j-1}^{1/(2j-1)}} (\log Q_j)^{16k} \right)^{(2j-1)/2j} x\left(\log\left(x^j e\right)\right)^{(j^2-1)/2j} Q_{j-1}^{-1/2j} (\log Q_j)^{16k}.$$

We deduce from (5.23) and (5.24)

$$\sum_{n\in[(9/10)x,x]} \left| \sum_{\substack{R/m<d<V/m,\ (d,m)=1 \\ d\in\mathcal{D}}} \phi^{-2}(d) A_1(d,n,l) \right| \tag{5.25}$$

$$\ll x Q_0^{-1/2} (\log x)^{16k} + x^{7/8} (\log x)^{32k} \sum_{j=2}^{L} \left(\log x^{j+1}\right)^{(j^2-1)/2j}.$$

For the sum in (5.25) we get for a sufficiently small d

$$\sum_{j=2}^{L} (\dots) \leqq \sum_{j=2}^{L} ((j+1)\log x)^{j/2} \leqq 2\frac{\log P}{\log\log x} \left(3\frac{\log P}{\log\log x} \log x \right)^{\frac{\log P}{\log\log x}} \ll P^3.$$

From this and (5.25) it follows together with the definition of Q_0, $m \leqq R^{1/3}$ and a sufficiently small d:

$$\sum_{n\in[(9/10)x,x[} \left| \sum_{\substack{R/m<d<V/m\ (d,m)=1 \\ d\in\mathcal{D}}} \phi^{-2}(d) A_1(d,n) \right| \tag{5.26}$$

$$\ll x(\log x)^{32k} \left(P^{-\frac{1}{2}\frac{2}{3}\frac{4}{5}} + P^3 x^{-1/8} \right) \ll x P^{-1/9}.$$

In order to finish the estimate of $F_2(n,R)$, we need the following result:

(5.27)
$$\sum_{\substack{m\leqq R^{1/3}\\ m\in\mathcal{D}}}\phi^{-2}(m)|A_2(m,n)|\leqq\prod_{\substack{p\leqq P\\ p|n}}\left(1+\frac{1}{p-1}\right)\prod_{\substack{p\leqq P\\ p\nmid n}}\left(1+\frac{1}{(p-1)^2}\right)\ll 2^{\omega(n)}.$$

Then (5.19), (5.26), (5.27) and $2^{\omega(n)}\leqq\tau(n)\ll_\epsilon n^\epsilon$ imply

$$\sum_{n\in[(9/10)x,x]}\sum_{i=1}^{s}F_2(n,P/l_i)\ll xP^{-1/10}. \tag{5.28}$$

So from the last expression, (5.19) and (5.20) we derive for all but $\ll x^{1-\delta_1}$ $n\in[(9/10)x,x[$, that satisfy the congruence conditions in (1.1):

$$T_1(n,P/l_i)\ll P^{-1/16} \tag{5.29}$$

for all l_i, $i\in\{1,\ldots,s\}$. By Lemma 4.3 (b) we get for $T_2(n,R)$ and $v=\dfrac{\log\log x}{2\log P}$:

$$T_2(n,R)\leqq\sum_{q\in\mathcal{D}}\left(\frac{q}{V}\right)^v\phi^{-2}(q)\,|A(q,n,l)|\leqq V^{-v}\prod_{p\leqq P}\left(1+p^v\frac{|A(p,n)|}{(p-1)^2}\right).$$

By
$$V^{-v}=x^{-1/2}$$
and
$$p^v\leqq(\log x)^{1/2},$$
it follows for a sufficiently small d:

$$\begin{aligned}T_2(n,R)&\leqq x^{-1/2}\prod_{p\leqq P}\left(1+\frac{4k(\log x)^{1/2}}{p}\right)\\&\ll x^{-1/2}(\log P)^{4k(\log x)^{1/2}}\ll x^{-1/3}.\end{aligned} \tag{5.30}$$

From (5.15), (5.29) and (5.30) the lemma follows.

LEMMA 5.6. (a) *For all n that satisfy the congruence conditions in* (1.1)

$$\prod_{p\leqq P}\left(1+\frac{A(p,n)}{(p-1)^2}\right)\gg(\log P)^{-2k}.$$

(b) *For any two primitive characters* $\chi_1 \pmod{q_1}$ *and* $\chi_2 \pmod{q_2}$, $q_3 = [q_1 q_2] \leqq P$ *and all* n, *which satisfy the congruence condition in* (1.1)

$$\left|\frac{A(q_3, n, \chi_1\chi_{0,q_3}, \chi_2\chi_{0,q_3})}{\phi^2(q_3)}\right| \prod_{\substack{p\leqq P\\(p,q_3)=1}} \left(1+\frac{A(p,n)}{(p-1)^2}\right) \ll \prod_{p\leqq P}\left(1+\frac{A(p,n)}{(p-1)^2}\right).$$

holds true.

PROOF. (a) By $0 \leqq w(n,p) \leqq (k, p-1)$ and Lemma 4.4 (d):

$$\prod_{p\leqq P}\left(1-\frac{(w(n,p)-1)p+1}{(p-1)^2}\right) \gg \prod_{2k<p\leqq P}\left(1-\frac{2k}{p}\right) \gg (\log P)^{-2k},$$

from which the lemma can be deduced by Lemma 4.4 (a).

(b) If $q_3 = 1$, the lemma is obvious. For $q_3 > 1$ we distinguish the cases (i) $q_1 = q_3$ and (ii) $1 \leqq q_1 < q_3$. In the case (i) we immediately get the desired result from Lemma 4.4 (a) and (b) by

$$\left|\frac{A(q_3, n, \chi_1\chi_{0,q_3}, \chi_2\chi_{0,q_3})}{\phi^{-2}(q_3)}\right| \prod_{\substack{p\leqq P\\(p,q_3)=1}} \left(1+\frac{A(p,n)}{(p-1)^2}\right) \leqq \prod_{p\leqq P}\left(1+\frac{A(p,n)}{(p-1)^2}\right).$$

(ii) Analogously to (4.2) we have only to take into consideration such pairs q_3 and q_1, for which $\left(\frac{q_3}{q_1}, q_1\right) = 1$ and so, by Lemma 4.3 (b),

$$A(q_3, n, \chi_1\chi_{0,q_3}, \chi_2\chi_{0,q_3}) = A(q_1, n, \chi_1, \chi_5) A\left(\frac{q_3}{q_1}, n, \chi_{0,\frac{q_3}{q_1}}, \chi_6\right),$$

for certain characters χ_5 and χ_6. Since in (4.6)

$$A\left(\frac{q_3}{q_1}, n, \chi_{0,\frac{q_3}{q_1}}, \chi_6\right) \ll \left(\frac{q_3}{q_1}\right)^{3/2+\epsilon},$$

furthermore, by Lemma 4.4 (a) and (d)

$$\prod_{p|\frac{q_3}{q_1}}\left(1+\frac{A(p,n)}{(p-1)^2}\right)^{-1} \ll \prod_{p|\frac{q_3}{q_1}, p>4k}\left(1-\frac{2k}{p}\right)^{-1} \leqq 2^{\omega\left(\frac{q_3}{q_1}\right)} \ll_\epsilon \left(\frac{q_3}{q_1}\right)^{\epsilon}.$$

Using all this we get together with the result from (i)

$$\left|\frac{A(q_3, n, \chi_1\chi_{0,q_3}, \chi_2\chi_{0,q_3})}{\phi^2(q_3)}\right| \prod_{\substack{p\leqq P\\(p,q_3)=1}} \left(1+\frac{A(p,n)}{(p-1)^2}\right) = \left|\frac{A(q_1, n, \chi_1, \chi_5)}{\phi^2(q_1)}\right|$$

$$\times \prod_{\substack{p\leqq P\\(p,q_1)=1}} \left(1+\frac{A(p,n)}{(p-1)^2}\right) \left|\frac{A\left(\frac{q_3}{q_1}, n, \chi_0, \frac{q_3}{q_1}, \chi_6\right)}{\phi^2\left(\frac{q_3}{q_1}\right)}\right| \prod_{p|\frac{q_3}{q_1}} \left(1+\frac{A(p,n)}{(p-1)^2}\right)^{-1}$$

$$\ll \prod_{p\leqq P} \left(1+\frac{A(p,n)}{(p-1)^2}\right) \phi^{-2}\left(\frac{q_3}{q_1}\right)\left(\frac{q_3}{q_1}\right)^{3/2+\epsilon}\left(\frac{q_3}{q_1}\right)^{\epsilon} \ll \prod_{p\leqq P}\left(1+\frac{A(p,n)}{(p-1)^2}\right).$$

6. The minor arcs

We obtain by Bessel's inequality and the prime number theorem

$$\sum_{(9/10)x\leqq n<x} r_2(x,n)^2 \leqq \int_m |S(\alpha)S_k(\alpha)|^2 d\alpha \ll x\log x \sup_{\alpha\in m} |S_k(\alpha)|^2.$$

By the definiton of the minor arcs and Theorem 1 in [4] we have

$$\sup_{\alpha\in m} |S_k(\alpha)| \ll x^{\frac{1+\epsilon}{k}} \left(\frac{1}{P}+\frac{1}{x^{1/2k}}+\frac{Q}{x}\right)^{1/4^{k-1}} \ll \frac{x^{\frac{1+\epsilon}{k}}}{P^{1/4^{k-1}}}.$$

Substituting this in the first estimate we obtain

$$\sum_{(9/10)x\leqq n<x} r_2(x,n)^2 \ll \frac{x^{1+(2/k)+\epsilon}}{P^{2/4^{k-1}}}. \tag{6.1}$$

7. The major arcs

Let us suppose in the following $l\in\{1,k\}$ and $S(\alpha)=S_1(\alpha)$. For $\alpha\in I(a,q)$ let $\alpha=\frac{a}{q}+\eta$. Because of $q\leqq P$ and $p>P$ for all p appearing in $S_l(\alpha)$ we get in a well-known way:

$$\begin{aligned} S_l(\alpha) &= \sum_{\frac{\sqrt[l]{x}}{2}\leqq p<\sqrt[l]{x}} \log p\, e\left(\frac{a}{q}p^l+\eta p^l\right) = \frac{1}{\phi(q)}\sum_{\chi \bmod q}\sum_{h=1}^{q}\overline{\chi}(h)e\left(\frac{ah^l}{q}\right) \\ &\times \sum_{\frac{\sqrt[l]{x}}{l}\leqq p<\sqrt[l]{x}} \chi(p)\log p\; e\left(\eta p^l\right) = \frac{1}{\phi(q)}\sum_{\chi \bmod q} C_l(\overline{\chi},a)S_l(\chi,\eta). \end{aligned} \tag{7.1}$$

Let $L=T$ if $l=1$ and $L=F$ if $l=k$. Now $W_l(\chi,\eta)$ is defined in the following way:

(i) For $\chi = \chi_{0,q}$ let

$$W_l(\chi, \eta) = S_l(\chi_{0,q}, \eta) - L(\eta) + \sum_{\substack{\varrho \in \theta' \cup \tilde{\beta} \\ \zeta(\varrho)=0}} L_\varrho(\eta).$$

(ii) For $\chi = \chi_{0,q}\chi^*$ with $\chi^* \in \theta \cup \tilde{\chi}$, $\chi^* \neq \chi_{0,1}$ let

$$W_l(\chi, \eta) = S_l(\chi_{0,q}\chi^*, \eta) + \sum_{\substack{\varrho \in \theta' \cup \tilde{\beta} \\ L(\varrho, \chi^*)=0}} L_\varrho(\eta).$$

(iii) In all other cases let

$$W_l(\chi, \eta) = S_l(\chi, \eta).$$

We obtain

$$S_l\left(\frac{a}{q} + \eta\right) = \frac{1}{\phi(q)} C_l(\chi_0, a) L(\eta) + \frac{1}{\phi(q)} D_l(a, q, \eta) + \frac{1}{\phi(q)} E_l(a, q, \eta),$$

where

$$D_l(a, q, \eta) = \sum_{\chi \bmod q} C_l(\overline{\chi}, a) W_l(\chi, \eta),$$

$$E_l(a, q, \eta) = -\sum_{\substack{\chi \in \theta \cup \tilde{\chi} \\ \mathrm{cond}\, \chi | q}} \sum_{\substack{\varrho \in \theta' \cup \tilde{\beta} \\ L(\varrho, \chi)=0}} C_l(\chi_{0,q}\overline{\chi}, a) L_\varrho(\eta).$$

Writing $W = W_1$, $E = E_1$ and $D = D_1$ we obtain from (3.6) and (7.1)

$$\begin{aligned}
r_1(x, n) &= \sum_{q \leqq P} \sum_{a=1}^{q}{}^{*} e\left(\frac{-an}{q}\right) \int_{-1/Q}^{1/Q} S\left(\frac{a}{q} + \eta\right) S_k\left(\frac{a}{q} + \eta\right) e(-n\eta) d\eta \\
&= \sum_{q \leqq P} \frac{1}{\phi^2(q)} A(q, n) \int_{-1/Q}^{1/Q} T(\eta) F(\eta) e(-n\eta) d\eta \\
&\quad + \sum_{q \leqq P} \frac{1}{\phi^2(q)} \sum_{a=1}^{q}{}^{*} e\left(-\frac{an}{q}\right) C_1(\chi_0, a) \int_{-1/Q}^{1/Q} T(\eta) D_k(a, q, \eta) e(-n\eta) d\eta \\
&\quad + \sum_{q \leqq P} \frac{1}{\phi^2(q)} \sum_{a=1}^{q}{}^{*} e\left(-\frac{an}{q}\right) C_1(\chi_0, a) \int_{-1/Q}^{1/Q} T(\eta) E_k(a, q, \eta) e(-n\eta) d\eta
\end{aligned}$$

$$+\sum_{q\leqq P}\frac{1}{\phi^2(q)}\sum_{a=1}^{q}{}^{*}e\left(-\frac{an}{q}\right)C_k(\chi_0,a)\int_{-1/Q}^{1/Q}F(\eta)D(a,q,\eta)e(-n\eta)d\eta$$

(7.2)

$$+\sum_{q\leqq P}\frac{1}{\phi^2(q)}\sum_{a=1}^{q}{}^{*}e\left(-\frac{an}{q}\right)C_k(\chi_0,a)\int_{-1/Q}^{1/Q}F(\eta)E(a,q,\eta)e(-n\eta)d\eta$$

$$+\sum_{q\leqq P}\frac{1}{\phi^2(q)}\sum_{a=1}^{q}{}^{*}e\left(-\frac{an}{q}\right)\int_{-1/Q}^{1/Q}D_k(a,q,\eta)E(a,q,\eta)e(-n\eta)d\eta$$

$$+\sum_{q\leqq P}\frac{1}{\phi^2(q)}\sum_{a=1}^{q}{}^{*}e\left(-\frac{an}{q}\right)\int_{-1/Q}^{1/Q}D_k(a,q,\eta)D(a,q,\eta)e(-n\eta)d\eta$$

$$+\sum_{q\leqq P}\frac{1}{\phi^2(q)}\sum_{a=1}^{q}{}^{*}e\left(-\frac{an}{q}\right)\int_{-1/Q}^{1/Q}D(a,q,\eta)E_k(a,q,\eta)e(-n\eta)d\eta$$

$$+\sum_{q\leqq P}\frac{1}{\phi^2(q)}\sum_{a=1}^{q}{}^{*}e\left(-\frac{an}{q}\right)\int_{-1/Q}^{1/Q}E(a,q,\eta)E_k(a,q,\eta)e(-n\eta)d\eta$$

$$=: S_1+S_2+S_3+S_4+S_5+S_6+S_7+S_8+S_9.$$

In the following we only take into consideration such $n\in[(9/10)x,x[$, that satisfy the congruence conditions in (1.1).

8. The calculation of S_1–S_9

We first estimate S_4. Changing the summation over the characters according to the inducing primitive characters, we get by Lemma 4.7 (a) and Cauchy's inequality:

$$S_4=\sum_{q\leqq P}\frac{1}{\phi^2(q)}\sum_{\chi \bmod q}A(q,n,\overline{\chi},\chi_0)\int_{-1/Q}^{1/Q}F(\eta)W(\chi,\eta)e(-n\eta)d\eta$$

(8.1)

$$\ll x^{(1/k)-(1/2)}\sum_{r\leqq P}\sum_{\chi \bmod r}{}^{*}\sum_{\substack{q\leqq P\\ q\equiv 0(\bmod r)}}\frac{1}{\phi^2(q)}\left|A(q,n,\overline{\chi}\chi_{0,q},\chi_{0,q})\right|$$

$$\times \left(\int_{-1/Q}^{1/Q} |W(\chi_{0,q}\chi, \eta)|^2 d\eta \right)^{1/2}.$$

Because of $q \leqq P$ and $p > P$ we have $W(\chi_{0,q}\chi, \eta) = W(\chi, \eta)$, and so we get by (8.1) and Lemma 4.5

$$\begin{aligned} S_4 &\ll x^{(1/k)-(1/2)} \sum_{r \leqq P} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/Q}^{1/Q} |W(\chi, \eta)|^2 d\eta \right)^{1/2} \\ &\quad \times \sum_{\substack{q \leqq P \\ q \equiv 0 (\bmod r)}} \frac{1}{\phi^2(q)} |A(q, n, \overline{\chi}\chi_{0,q}, \chi_{0,q})| \qquad (8.2) \\ &\ll \log^{5k+1} x \sum_{r \leqq P} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/Q}^{1/Q} |W(\chi, \eta)|^2 d\eta \right)^{1/2}. \end{aligned}$$

We define now for an arbitrary primitive character $\chi \bmod r$:

$$\sideset{}{^\#}\sum_{t}^{t+h} \chi(p) \log p = \begin{cases} \sum_{t}^{t+h} \log p - \sum_{t}^{t+h} 1 & \text{if } r = 1, \\ \sum_{t}^{t+h} \chi(p) \log p & \text{if } r > 1. \end{cases}$$

Then we get by Lemma 1 in [3] and the definition of $W(\chi, \eta)$:

$$\int_{-1/qQ}^{1/qQ} |W(\chi, \eta)|^2 d\eta \ll \int_{\frac{x}{4}}^{x} \left| \frac{1}{Q} \sideset{}{^\#}\sum_{\substack{t \leqq p \leqq t + \frac{Q}{2}, \\ \frac{x}{2} \leqq p < x}} \chi(p) \log p \right|^2 dt,$$

from which we get by (8.2):

$$\begin{aligned} S_4 &\ll x^{1/k} \log^{5k+1} x \sum_{r \leqq P} \sideset{}{^*}\sum_{\chi \bmod r} \max_{x/4 \leqq t \leqq x} \max_{h \leqq xP^{-4k-3}} (h + xP^{-4k-3})^{-1} \qquad (8.3) \\ &\quad \times \left| \sideset{}{^\#}\sum_{t}^{t+h} \chi(p) \log p \right|. \end{aligned}$$

Arguing exactly as in (19) in [1] we obtain for the last double sum

$$\sum \sum \ll \delta^{8k^2+1} \log^{-8k^2} x + P^{-1}. \qquad (8.4)$$

If we combine (8.3) and (8.4) and argue in the same way for S_8, where we use the upper (3.1) for the number of the *P-exceptional zeros* over which is summed in S_8, we obtain

$$S_4 + S_8 \ll \frac{\delta^{8k^2+1} x^{1/k}}{\log^{8k^2-5k-1,5} x} + \frac{x^{1/k} \log^{5k+1,5} x}{P}. \tag{8.5}$$

Using Lemma 4.7 (b) we get in the same way for S_7:

$$\begin{aligned} S_7 = \sum_{q \leqq P} \frac{1}{\phi^2(q)} \sum_{\chi \bmod q} \sum_{\chi_1 \bmod q} A(q, n, \overline{\chi_1}, \overline{\chi_2}) \int_{-1/Q}^{1/Q} W(\chi, \eta) W_k(\chi_1, \eta) e(-n\eta) d\eta \\ \ll \left(\delta^{8k^2+1} x^{1/2} \log^{5k+1-8k^2} x + \frac{x^{1/2} \log^{5k+1} x}{P} \right) \\ \times \sum_{r_1 \leqq P} \sum_{\chi_1 \bmod r_1}{}^{*} \left(\int_{-1/Q}^{1/Q} |W_k(\chi_1, \eta)|^2 d\eta \right)^{1/2}. \end{aligned} \tag{8.6}$$

Arguing as in (8.3) and (8.4) we derive from this

$$S_7 \ll \left(\delta^{8k^2+1} x^{1/k} \log^{5k+1-8k^2} x + \frac{x^{1/k} \log^{5k+1} x}{P} \right) W_k, \tag{8.7}$$

where

$$\begin{aligned} W_k = \sum_{r_1 \leqq P} \sum_{\chi_1 \bmod r_1}{}^{*} \max_{\sqrt[k]{x/(2^{k+1})} \leqq y \leqq \sqrt[k]{x}} \max_{h \ll x^{1/k} P^{-4k-3}} (h + \sqrt[k]{x} P^{-4k-3})^{-1} \\ \times \left| \sum_{y}^{y+h}{}^{\#} \chi_1(p) \log p \right|, \end{aligned}$$

and

$$W_k \ll \delta^{8k+1} \log^{-8k} x + P^{-1}. \tag{8.8}$$

Combining (8.7) and (8.8) and arguing in the same way for S_2 and S_6 by using again (3.1) we obtain

$$S_2 + S_6 + S_7 \ll \frac{\delta^{8k+1} x^{1/k}}{\log^{3k-1,5} x} + \frac{x^{1/k} \log^{5k+1,5} x}{P}. \tag{8.9}$$

For S_1 we get by the Lemmas 4.5 and 4.9

(8.10)
$$\begin{aligned} S_1 &= \sum_{q \leqq P} \frac{1}{\phi^2(q)} A(q,n) \int_0^1 T(\eta)F(\eta)e(-n\eta)d\eta \\ &+ O\left(\sum_{q \leqq P} \frac{1}{\phi^2(q)} |A(q,n)| \int_{1/Q}^{1/2} |T(\eta)F(\eta)|d\eta \right) \\ &= \sigma(n,P)L(x,n) + O\left(x^{1/k} P^{\frac{-2k}{s}}\right). \end{aligned}$$

Noting that in the sum defining S_3 by (4.2) we only have to take into consideration such q with l cond $\chi = q$, for which $(l,\ \text{cond}\ \chi) = 1$ holds, we obtain in the same way as for S_1:

(8.11)
$$\begin{aligned} S_3 = &- \sum_{r \leqq P} \sum_{\substack{\chi \in \theta \cup \tilde{\chi} \\ \chi \bmod r}} \sum_{\substack{\varrho \in \theta' \cup \tilde{\beta} \\ L(\varrho,\chi)=0}} \frac{1}{\phi^2(r)} A(r,n,\chi_{0,r},\overline{\chi}) \sigma\left(n, \frac{P}{r}, r\right) L_{1,\varrho}(x,n) \\ &+ O\left(x^{1/k} P^{\frac{-2k}{s}}\right). \end{aligned}$$

For the calculation of the remaining terms we define

$$\theta_1' = \left\{ \varrho \in \theta' \cup \tilde{\beta} :\ |\gamma| \leqq P^{4k+3} \right\}, \qquad \theta_2' = \theta' \cup \tilde{\beta} \setminus \theta_1',$$

such that by (3.2):

(8.12)
$$\varrho = \beta + i\gamma \in \theta_2' \Longrightarrow |\gamma| > 16P^{4k+3}.$$

So we obtain
(8.13)
$$\begin{aligned} S_5 = &- \sum_{\chi \in \theta \cup \tilde{\chi}} \sum_{\substack{\varrho \in \theta_1' \\ L(\varrho,\chi)=0}} \sum_{\substack{q \leqq P \\ \text{cond}\ \chi | q}} \frac{1}{\phi^2(q)} A(q,n,\overline{\chi}\chi_{0,q},\chi_{0,q}) \int_{-1/Q}^{1/Q} T_\varrho(\eta)F(\eta)e(-n\eta)d\eta \\ &- \sum_{\chi \in \theta} \sum_{\substack{\varrho \in \theta_2' \\ L(\varrho,\chi)=0}} \sum_{\substack{q \leqq P \\ \text{cond}\ \chi | q}} \frac{1}{\phi^2(q)} A(q,n,\overline{\chi}\chi_{0,q},\chi_{0,q}) \int_{-1/Q}^{1/Q} T_\varrho(\eta)F(\eta)e(-n\eta)d\eta \\ &=: S_{5,1} + S_{5,2}. \end{aligned}$$

We first get from Lemma 4.5, Lemma 4.10 and (3.1)

(8.14)
$$S_{5,2} \leqq \sum_{\chi\in\theta} \sum_{\substack{\varrho\in\theta'_2 \\ L(\varrho,\chi)=0}} \sum_{\substack{q\leqq P \\ \mathrm{cond}\,\chi|q}} \frac{1}{\phi^2(q)} |A(q,n,\overline{\chi}\chi_{0,q},\chi_{0,q})| \times \int_{-1/Q}^{1/Q} |T_\varrho(\eta)F(\eta)|d\eta \ll x^{1/k}P^{-2k}.$$

Arguing as for S_3, we get by appealing again to (3.1)

(8.15)
$$S_5 = -\sum_{r\leqq P} \sum_{\substack{\chi\in\theta\cup\tilde{\chi} \\ \chi \bmod r}} \sum_{\substack{\varrho\in\theta'_1 \\ L(\varrho,\chi)=0}} \frac{1}{\phi^2(r)} A(r,n,\overline{\chi},\chi_{0,r}) \times \sigma\left(n,\frac{P}{r},r\right) L_{\varrho,1}(x,n) + O\left(x^{1/k}P^{\frac{-2k}{s}}\right).$$

For S_9 we get similarly to S_5

(8.16)
$$S_9 = \sum_{\chi\in\theta\cup\tilde{\chi}} \sum_{\substack{\varrho\in\theta'_1 \\ L(\varrho,\chi)=0}} \sum_{\chi_1\in\theta\cup\tilde{\chi}} \sum_{\substack{\varrho'\in\theta'\cup\tilde{\beta} \\ L(\varrho',\chi_1)=0}} \sum_{\substack{r\leqq P \\ [\mathrm{cond}\,\chi,\,\mathrm{cond}\,\chi_1]=r}} \frac{1}{\phi^2(r)} A(r,n,\overline{\chi}\chi_{0,r},\overline{\chi_1}\chi_{0,r}) \times \sigma\left(n,\frac{P}{r},r\right) L_{\varrho,\varrho'}(x,n) + O\left(x^{1/k}P^{\frac{-2k}{s}}\right).$$

9. Proof of the theorem

We first notice that obviously

(9.1)
$$|L_{\varrho,\varrho'}(X,n)| = \Big|\sum_{\substack{n-x<m^k\leqq n-(x/2) \\ \frac{\sqrt[k]{x}}{2}\leqq m<\sqrt[k]{x}}} (n-m^k)^{\varrho-1}m^{\varrho'-1}\Big| \ll x^{1/k}x^{\beta-1}x^{\beta'-1}.$$

Arguing in exactly the same way as in (35) in [1] or in Lemma 2.1 in [8], we obtain further that

(9.2)
$$\sum_{\varrho\in\theta'} x^{\beta-1} + \sum_{\varrho\in\theta'}\sum_{\varrho'\in\theta'} x^{\beta-1}x^{\beta'-1} \leqq c_6 \exp\left(-\frac{c_1}{2b}\right)\delta^2 + x^{-1/2},$$

where in the sequel we will neglect $x^{1/2}$, which in (9.10) will be shown to be permissible. We define further

$H = \{r = [r_1r_2],\ r_i = P\text{-excluded module or exceptional module to } P \text{ or } 1\}$,
$G = \{r \in H,\ r \geqq P^{1/5}\}$.

Then we derive from Lemmas 5.3 and 5.4 and (9.1) that for any two characters $\chi_1 (\bmod r_1), \chi_2 (\bmod r_2) \in \{\theta \cup \tilde{\chi} \cup \chi_{0,1}\}$ with $r = [r_1, r_2] =, r \in G$,

$$\frac{1}{\phi^2(r)} |A(r, n, \chi_1 \chi_{0,r}, \chi_2 \chi_{0,r})| |\sigma\left(n, \frac{P}{r}, r\right)| |L_{\varrho,\varrho'}(x, n)| \ll x^{1/k} P^{-1/240}, \tag{9.3}$$

holds for all but $\ll x P^{-1/80}$ $n \in [(9/10)x, x[$. If – in view of (3.1) – we apply Lemma 5.5 to all $r \in H \setminus G$ and note that $\tilde{r} \notin G$ for a sufficiently small λ, then for all $n \in [(9/10)x, x[$ that satisfy the congruence conditions in (1.1) and $n \notin A(x)$ with $|A(x)| \ll x P^{-1/80} (\log x)^{1/3} + x^{1-\delta_1} \ll x^{1-\delta_2}$, $\delta_2 \geqq 0$, there holds by (7.2), (8.5), (8.9), (8.10), (8.11), (8.15) and (8.16):

$$\begin{aligned} r_1(x,n) = {} & \prod_{p \leqq P} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) L(x,n) \\ & - \frac{1}{\phi^2(\tilde{r})} A(\tilde{r}, n, \tilde{\chi}, \chi_{0,\tilde{r}}) \prod_{\substack{p \leqq P \\ (p,\tilde{r})=1}} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) L_{\tilde{\beta},1}(x,n) \\ & - \frac{1}{\phi^2(\tilde{r})} A(\tilde{r}, n, \chi_{0,\tilde{r}}, \tilde{\chi}) \prod_{\substack{p \leqq P \\ (p,\tilde{r})=1}} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) L_{1,\tilde{\beta}}(x,n) \\ & + \frac{1}{\phi^2(\tilde{r})} A(\tilde{r}, n, \tilde{\chi}, \tilde{\chi}) \prod_{\substack{p \leqq P \\ (p,\tilde{r})=1}} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) L_{\tilde{\beta},\tilde{\beta}}(x,n) \\ & - \sum_{\substack{r \leqq P \\ r \notin G}} \sum_{\substack{\chi \in \theta \\ \chi \bmod r}} \sum_{\substack{\varrho \in \theta'_1 \setminus \beta \\ L(\varrho,\chi)=0}} \frac{1}{\phi^2(r)} A(r, n, \overline{\chi}, \chi_{0,r}) \prod_{\substack{p \leqq P \\ (p,r)=1}} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) L_{\varrho,1}(x,n) \\ & - \sum_{\substack{r \leqq P \\ r \notin G}} \sum_{\substack{\chi \in \theta \\ \chi \bmod r}} \sum_{\substack{\varrho \in \theta' \\ L(\varrho,\chi)=0}} \frac{1}{\phi^2(r)} A(r, n, \chi_{0,r}, \overline{\chi}) \prod_{\substack{p \leqq P \\ (p,r)=1}} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) L_{1,\varrho}(x,n) \\ & \sum_{\chi \in \theta \cup \tilde{\chi}} \sum_{\substack{\varrho \in \theta'_1 \\ L(\varrho,\chi)=0}} \sum_{\chi_1 \in \theta \cup \tilde{\chi}} \sum_{\substack{\varrho' \in \theta' \cup \tilde{\beta} \\ L(\varrho',\chi)=0 \\ (\varrho,\varrho') \neq (\tilde{\beta},\tilde{\beta})}} \sum_{\substack{r \leqq P,\, r \notin G, \\ [\operatorname{cond} \chi,\, \operatorname{cond} \chi_1] = r}} \frac{1}{\phi^2(r)} A(r, n, \overline{\chi} \chi_{0,r}, \overline{\chi_1} \chi_{0,r}) \\ & \times \prod_{\substack{p \leqq P \\ (p,r)=1}} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) L_{\varrho,\varrho'}(x,n) + O(\dots) \\ = {} & B_1 + \cdots + B_7 + O(x^{1/k} P^{\frac{-2k}{s}} + x^{1/k} \delta^2 \log^{1,5-3k} x), \end{aligned} \tag{9.4}$$

where we have used (3.1) for the calculation of the error term. In the following s will be chosen fixed according to the preceding discussion. We first get by (9.1), (9.2) and Lemma 5.6 (b)

$$B_5+\cdots+B_7 \ll x^{1/k}\prod_{p\leqq P}\left(1+\frac{A(p,n)}{(p-1)^2}\right)\left(\sum_{\varrho\in\theta'}x^{\beta-1}+\sum_{\varrho\in\theta'}\sum_{\varrho'\in\theta'}x^{\beta-1}x^{\beta'-1}\right)$$

$$\leqq c_7\exp\left(\frac{-c_1}{2b}\right)\delta^2 x^{1/k}\left|\prod_{p\leqq P}\left(1+\frac{A(p,n)}{(p-1)^2}\right)\right|. \tag{9.5}$$

We further derive from Lemma 4.1 (c) and (d) that

$$\prod_{p\leqq P}\left(1+\frac{A(p,n)}{(p-1)^2}\right)=\prod_{\substack{p\leqq P\\(p,\tilde r)=1}}\left(1+\frac{A(p,n)}{(p-1)^2}\right)\frac{\tilde r}{\phi^2(\tilde r)}\sum_{\substack{l+m^k\equiv n(\bmod \tilde r)\\1\leqq l,m\leqq\tilde r,(lm,\tilde r)=1}}1. \tag{9.6}$$

In the same way as in the proof of Lemma 4.4 (b) we obtain for the characters $\chi_1,\chi_2\in\{\chi_{0,\tilde r},\tilde\chi\}$, which are not both equal to $\chi_{0,\tilde r}$:

$$A(\tilde r,n,\chi_1,\chi_2)=\tilde r\sum_{\substack{l+m^k\equiv n(\bmod \tilde r)\\1\leqq l,m\leqq\tilde r,(lm,\tilde r)=1}}\chi_1(l)\chi_2(m). \tag{9.7}$$

So we get from (9.4), (9.6) and (9.7)

$$\begin{aligned}
&B_1+B_2+B_3+B_4\\
&=\prod_{\substack{p\leqq P\\(p,\tilde r)=1}}\left(1+\frac{A(p,n)}{(p-1)^2}\right)\frac{\tilde r}{\phi^2(\tilde r)}\\
&\times\Bigg((L(x,n))\sum_{\substack{l+m^k\equiv n(\bmod \tilde r)\\1\leqq l,m\leqq\tilde r,(lm,\tilde r)=1}}1-L_{\tilde\beta,1}(x,n)\sum_{\substack{l+m^k\equiv n(\bmod \tilde r)\\1\leqq l,m\leqq\tilde r,(lm,\tilde r)=1}}\tilde\chi(l)\\
&-L_{1,\tilde\beta}(x,n)\sum_{\substack{l+m^k\equiv n(\bmod \tilde r)\\1\leqq l,m\leqq\tilde r,(lm,\tilde r)=1}}\tilde\chi(m)+L_{\tilde\beta,\tilde\beta}(x,n)\sum_{\substack{l+m^k\equiv n(\bmod \tilde r)\\1\leqq l,m\leqq\tilde r,(lm,\tilde r)=1}}\tilde\chi(l)\tilde\chi(m)\Bigg)\\
&=\prod_{\substack{p\leqq P\\(p,\tilde r)=1}}\left(1+\frac{A(p,n)}{(p-1)^2}\right)\frac{\tilde r}{\phi^2(\tilde r)} \qquad (9.8)\\
&\times\left(\sum_{\substack{a+b^k=n\\ \frac{x}{2}\leqq a<x\\ \frac{\sqrt[k]{x}}{2}\leqq b<\sqrt[k]{x}}}\sum_{\substack{l+m^k\equiv n(\bmod \tilde r)\\1\leqq l,m\leqq\tilde r,(lm,\tilde r)=1}}(1-\tilde\chi(l)a^{\tilde\beta-1})(1-\tilde\chi(m)b^{\tilde\beta-1}\right)
\end{aligned}$$

$$\geqq \left| \prod_{p \leqq P} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) \right| \sum_{\substack{a+b^k=n \\ \frac{x}{2} \leqq a < x \\ \frac{\sqrt[k]{x}}{2} \leqq b < \sqrt[k]{x}}} (1 - a^{\tilde{\beta}-1})(1 - b^{\tilde{\beta}-1}),$$

where in the last step we have again argued as in (9.7). If the Siegel zero $\tilde{\beta}$ exists, we get

$$1 - P^{\tilde{\beta}-1} = (1 - \tilde{\beta}) \log P \; P^{\gamma - 1} \geqq c_{14}(1 - \tilde{\beta}) \log P = c_{14}\delta.$$

Applying this to (9.8) we obtain

$$B_1 + B_2 + B_3 + B_4 \geqq \delta^2 x^{1/k} \left| \prod_{p \leqq P} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) \right|,$$

which, by (9.8), obviously also holds if $\tilde{\beta}$ does not exist. So we get for a sufficiently small b from the last inequality, (9.5) and Lemma 5.6 (a)

$$(9.9) \quad |B_1 + \cdots + B_7| \gg \delta^2 x^{1/k} \left| \prod_{p \leqq P} \left(1 + \frac{A(p,n)}{(p-1)^2}\right) \right| \frac{1}{2} c_7 \gg \delta^2 x^{1/k} \log^{-2k} x.$$

If $\tilde{\beta}$ exists, we know by Lemma 3.1 and (3.4):

$$(9.10) \qquad \delta^2 = ((1 - \tilde{\beta}) \log P)^2 \gg \frac{1}{P^{(4k+3)\lambda/(4k+2)} \log^2 x}.$$

Otherwise $\delta = 1$. We derive from (9.4), (9.9) and (9.10) that for $\lambda \leqq \min\left(\frac{1}{4^{k+1}}, \frac{k}{s}\right)$, $n \in [(9/10)x, x[\setminus A(x)$ and n satisfies the congruence conditions in (1.1):

$$r_1(x,n) \gg x^{1/k} \delta^2 \log^{-2k} x.$$

We further conclude from (6.1) that

$$r_2(x,n) \ll x^{1/k} P^{-1/4^k}$$

for all but $n \in [(9/10)x, x[\setminus B(x)$ with $|B(x)| \ll x^{1+\epsilon} P^{-6/4^k}$. So we get from (3.6) and the upper bound for λ

$$r(x,n) \gg x^{1/k} \delta^2 \log^{-2k} x$$

for all but $|A(x) \cup B(x)| \ll x^{1-\Theta}$, $\Theta > 0$ integers $n \in [(9/10)x, x[$, that satisfy the congruence conditions in (1.1). Splitting the interval $[1, x[$ into intervals of the type $[\frac{9}{10}t, t[$, we get the theorem.

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REFERENCES

[1] BRÜNNER, R., PERELLI, A. and PINTZ, J., The exceptional set for the sum of a prime and a square, *Acta Math. Hungar.* **53** (1989), 347–365. *MR* **91b**:11104

[2] DAVENPORT, H., *Multiplicative number theory*, Second edition, Revised by Hugh L. Montgomery, Graduate Texts in Mathematics, 74, Springer-Verlag, New York – Berlin, 1980. *MR* **82m**:10001

[3] GALLAGHER, P. X., A large sieve density estimate near $\sigma = 1$, *Invent. Math.* **11** (1970), 329–339. *MR* **43** #4775

[4] HARMAN, G., Trigonometric sums over primes, *Mathematika* **28** (1981), 249–254. *MR* **83j**:10045

[5] IRELAND, K. and ROSEN, M., *A classical introduction to modern number theory*, Graduate Texts in Mathematics, 84, Springer-Verlag, New York – Berlin, 1982. *MR* **83g**:12001

[6] LEUNG, M. C. and LIU M. C., On generalized quadratic equations in three prime variables, *Monatsh. Math.* **115** (1993), 133–167. *MR* **94g**:11085

[7] CHIU, S. F. and LIU, M. C., On exceptional sets for numbers representable by binary sums, *Rocky Mountain J. Math.* **26** (1996), 959–986. *MR* **98c**:11107

[8] LIU, M. C. and TSANG, K. M., Small prime solutions of some additive equations, *Monatsh. Math.* **111** (1991), 147–169. *MR* **92e**:11106

[9] MONTGOMERY, H. L. and VAUGHAN, R. C., The exceptional set in Goldbach's problem, *Acta Arith.* **27** (1975), 353–370. *MR* **51** #10263

[10] SCHMIDT, W. M., *Equations over finite fields. An elementary approach*, Lecture Notes in Mathematics, Vol. 536, Springer–Verlag, Berlin – New York, 1976. *MR* **55** #2744

[11] TITCHMARSH, E. C., *The theory of the Riemann zeta-function*, Second edition, Edited and with a preface by D. R. Heath-Brown, Oxford, Clarendon Press, 1986. *MR* **88c**:11049

[12] VAUGHAN, R. C., *The Hardy–Littlewood method*, Cambridge Tracts in Mathematics, 80, Cambridge University Press, Cambridge – New York, 1981. *MR* **84b**:10002

[13] VINOGRADOV, I. M., On the estimations of some simplest trigonometrical sums involving prime numbers, *Bull. Acad. Sci. URSS Sér. Math.* [*Izvestia Akad. Nauk SSSR*] **1939**, 371–396 and engl. summary 396–398 (in Russian). *Zbl* **24**, 293–294; *MR* **2**, 40

[14] ZACCAGNINI, A., On the exceptional set for the sum of a prime and a k-th power, *Mathematika* **39** (1992), 400–421. *MR* **94g**:11086

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MATHEMATISCHES INSTITUT
ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG
ALBERTSTRASSE 23 B
D–79104 FREIBURG
GERMANY

Claus.Bauer@mn.oen.siemens.de

Current address:

IM ACKER 18
D–56332 OBERFELL
GERMANY

clausbauer@yahoo.com