Acta Mathematica Sinica, New Series
1998, April, Vol.14, No.2, pp. 223–234

On a Problem of the Goldbach-Waring Type

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Abstract Let $E(X) = |\{N \le X; N \ne p_1^2 + p_2^3 + p_3^4 + p_4^5$ for any primes $p_i\}|$. It is proved in this paper that there exists a positive constant $\delta > 0$ such that

$$E(X) \ll_\delta X^{1-\delta}$$

which improves a result of prachar.

Keywords Goldbach-Waring problem, Exponential sum, Exceptional zero
1991MR Subject Classification 11P32, 11L07
Chinese Library Classification O156.

1 Introduction and Statement of Results

After 1937 I. M. Vinogradov[1] successfully proved the ternary Goldbach-conjecture, and its method was applied to similar problems in additive prime number theory by several mathematicians. Among them were Hua[2] and Prachar. The latter established in 1952 [3] the following result:

There exists a constant $c > 0$ such that all but $x(\log x)^{-c}$ even integers N smaller than x are representable as

$$N = p_1^2 + p_2^3 + p_3^4 + p_5^4 \tag{1.1}$$

for prime numbers p_i. We will improve on this result by establishing the following theorem:

Theorem *There exists a positive number δ such that all but $\ll x^{1-\delta}$ positive even integers $N \le x$ are representable as in* (1.1).

For the proof of the theorem we will apply a modification of the method of Montgomery and Vaughan[4] by Leung and Liu[5] and by Liu and Tsang[6] separately.

2 Notation and Structure of the Proof

We will choose our notation similar to that in [5]. By k we will always denote an integer $k \in \{2, 3, 4, 5\}$, and by p we denote a prime number. $c_1, c_2, \cdots$ are effective positive constants.

Received July 1, 1996, Accepted March 10, 1997
During the preparation of this article the author was staying at the Department of Mathematics at Shandong University, P.R.China. He was holding a joint scholarship by the Chinese State Education Commission and the German Academic Exchange Service (DAAD).

c and ϵ are positive constants which take different values at different places. δ shall denote a small positive number, which will be specified later, and we denote by $\omega(n)$ the number of prime divisors of n. Further put $P = N^{\delta_1}$, $T = P^{1/\sqrt{\delta_1}}$, $Q = NT^{-1/4}$ and

$$\mu = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - 1. \tag{2.1}$$

It is well-known (see [7]) that there is at most one primitive character to a modulus $q \le T$ for which the corresponding L-function has a zero in the region

$$\sigma > 1 - \eta(T), \quad |t| \le T, \quad \text{where} \quad \eta(T) = \frac{c_1}{\log T}, \tag{2.2}$$

for a small constant c_1. If there is such an exceptional character, it is real and we denote it by $\tilde{\chi}$. The corresponding exceptional zero, denoted by $\tilde{\beta}$, is real, simple and unique. In case $\tilde{\chi}$ exists, the zero-free region in (2.2) is widened to (see [8])

$$\eta(T) = \frac{c_2}{\log T} \log\left(\frac{ec_1}{(1-\tilde{\beta})\log T}\right). \tag{2.3}$$

For the exceptional module $\tilde{r}$ it is further known that

$$\frac{c_3}{\tilde{r}^{1/2}\log^2 \tilde{r}} \le 1 - \tilde{\beta} \le \frac{c_1}{\log T}. \tag{2.4}$$

We define for any $x > N^{1/2k}$ and any χ mod q with $q \le P$: $S_\chi(x,T) = \sum\limits_{|\gamma| \le T} x^{\beta - 1}$, where $\sum\limits_{|\gamma|\le T}$ denotes the summation over all zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ lying inside the region: $|\gamma| \le T, \frac{1}{2} \le \beta \le 1 - \eta(T)$, and $\eta(T)$ is defined in (2.3) or (2.2) according as $\tilde{\beta}$ exists or not. Let

$$\Omega = \begin{cases} (1-\tilde{\beta})\log T, & \text{if } \tilde{\beta} \text{ exists,} \\ 1, & \text{otherwise.} \end{cases} \tag{2.5}$$

Following the proof of Lemma 2.1 in [6], by appealing to Gallagher's density estimate ([8]) we can prove the following lemma:

Lemma 2.1 *If $x \ge N^{1/2k}$, there exists an absolute constant c_4 such that for a sufficiently small δ_1 $\sum\limits_{q \le P} \sum\limits_{\chi \bmod q} {}^* S_\chi(x,T) \ll \Omega^4 \exp(-c_4/\delta_1)$, where $\sum\limits_{\chi \bmod q}{}^*$ denotes the summation over all primtive characters $\chi(\mathrm{mod} q)$.*

Furthermore we define $M = \frac{N}{P^{\xi/16}}$, $M_k = M^{1/k}$, $N_k = N^{1/k}$, where $\xi = 4^{-5}$, and we use this to define $S_k(\alpha) = \sum\limits_{M_k \le n \le N_k} \Lambda(n) e(n^k \alpha)$, $S_k(\chi,\alpha) = \sum\limits_{M_k \le n \le N_k} \Lambda(n)\chi(n)e(n^k\alpha)$, for every character (mod q) with $q \le P$. $I_k(\alpha) = \int_{M_k}^{N_k} e(x^k\alpha)dx$, $\tilde{I}_k(\alpha) = \int_{M_k}^{N_k} x^{\tilde{\beta}-1} e(x^k\alpha)dx$, and $I_k(\chi,\alpha) = \int_{M_k}^{N_k} e(x^k\alpha) \sum\limits_{|\gamma|\le T} x^{\rho-1} d$. Let for any character χ mod q, $C_k(\chi, m) = \sum\limits_{l=1}^{q} \chi(l) e\left(\frac{ml^k}{q}\right)$.

We now define the major arcs M and minor arcs m as follows:

$$M = \sum_{q \le P} \sum_{a=1}^{q}{}^* I(a,q), \quad I(a,q) = \left[\frac{a}{q} - \frac{1}{Qq}, \frac{a}{q} + \frac{1}{Qq}\right], \quad m = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \Big\backslash M,$$

where

$$\sum_{a=1}^{q}{}^{*} = \sum_{\substack{a=1\\(a,q)=1}}^{q}.$$

Let

$$I(N) = \sum_{\substack{M_k \le n_k \le N_k, k\in\{2,\cdots,5\}\\ n_2^2+\cdots+n_5^5=N}} \Lambda(n_2)\cdots\Lambda(n_5).$$

Then we find

$$I(N) = \int_{\frac{1}{Q}}^{1+\frac{1}{Q}} e(-N\alpha)\prod_{k=2}^{5} S_k(\alpha)d\alpha \;= \left(\int_M + \int_m\right) e(-N\alpha)\prod_{k=2}^{5} S_k(\alpha)d\alpha =: I_1(N) + I_2(N). \tag{2.6}$$

3 Simplification of $I_1(N)$

For any α in $I(a,q)$ we have $\alpha = \frac{a}{q}+\eta, |\eta| \le \frac{1}{qQ}$. In the usual way we find

$$S_k(\alpha) = \phi^{-1}(q)\sum_{\chi \bmod q} C_k(\overline{\chi}, a)S_k(\chi,\eta), \tag{3.1}$$

where we have used that $p < q$ for all p and all q under consideration. We will now follow the arguments in [6] in order to simplify the contribution of the major arcs. The proofs will often be omitted because the respective lemmas can be shown in the same way as Lemmas 3.1 to 3.4 in [6].

Lemma 3.1 *For any real α and any χ mod q with $q \le T$, we have*

$$S_k(\chi,\eta) = \delta_{\chi_0} I_k(\alpha) - \delta_{\tilde\chi}\tilde I_k(\alpha) - I_k(\chi,\alpha) + O((1+|\eta|N)N^{1/k}L^2T^{-1}),$$

where

$$L = \log N, \quad \delta_{\chi_0} = \begin{cases} 1, & \text{if } \chi = \chi_0 (\text{mod } q),\\ 0, & \text{otherwise.}\end{cases}, \quad \delta_{\tilde\chi} = \begin{cases} 1, & \text{if } \chi = \tilde\chi\chi_0 (\text{mod } q),\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 3.2 *Let $\rho = \beta + i\gamma, 1/2 \le \beta \le 1$. Then for any real η we have*

$$\int_{M_k}^{N_k} e(x^k\eta)x^{\rho-1}dx \ll \begin{cases} \min(N_k^\beta, |\eta|^{-\frac{\beta}{k}+\frac{1}{k}-1}M_k^{1-k}), & \text{if } \gamma = 0,\\ N_k^\beta|\gamma|^{-1}, & \text{if } |\eta| \le \dfrac{|\gamma|}{4k\pi N_k^k},\\ N_k^k M_k^{\beta-k}|\gamma|^{-1/2}, & \text{if } \dfrac{|\gamma|}{4k\pi N_k^k} \le |\eta| \le \dfrac{|\gamma|}{k\pi M_k^k},\\ M_k^{\beta-k}|\eta|^{-1}, & \text{if } \dfrac{|\gamma|}{k\pi M_k^k} < |\eta|.\end{cases}$$

Lemma 3.3 *For any real η we have*

$$I_k(\eta) \ll \min(N_k, |\eta|^{-1}M_k^{1-k}), \quad \tilde I_k(\eta) \ll \min(N_k^{\tilde\beta}, |\eta|^{\frac{1-\tilde\beta}{k}}M_k^{1-k}),$$

$$I_k(\chi,\eta) \ll \begin{cases} N_k, & \text{for any real } \eta, \\ N_k^k M_k^{1-\frac{3k}{2}} |\eta|^{-1/2}, & \text{for } N_k^{-k} < |\eta| \le \frac{T}{k\pi M_k^k}, \\ M_k^{1-k} |\eta|^{-1}, & \text{for } \frac{T}{k\pi M_k^k} < |\eta|. \end{cases}$$

Lemma 3.4 *For $m \in \{1,2\}$ we have*

$$\int_{-\infty}^{\infty} |I_k(\eta)|^{2m} d\eta \ll \frac{N_k^{k(2m-1)}}{M_k^{(k-1)2m}}, \quad \int_{-\infty}^{\infty} |\tilde{I}_k(\eta)|^{2m} d\eta \ll \frac{N_k^{k(m(\tilde{\beta}+1)-1)}}{M_k^{(k-1)2m}},$$

$$\int_{-\infty}^{\infty} |I_k(\chi,\eta)|^{2m} d\eta \ll \frac{N_k^{3mk-k}}{M_k^{3mk-2m}}.$$

Proof The first three lemmas are proved in the same way as Lemmas 3.1–3.3 in [6]. The first estimate in Lemma 3.4 follows from Lemma 3.3 if we split up the integral in the following way:

$$\int_{-\infty}^{\infty} |I_k(\eta)|^{2m} d\eta \ll \int_{|\eta| \le N_k^{-k}} N_k^{2m} d\eta + \int_{N_k^{-k} < |\eta|} |\eta|^{-2m} M_k^{(1-k)2m} d\eta \ll \frac{N_k^{k(2m-1)}}{M_k^{(k-1)2m}}.$$

The second estimate is proved in the same way, whereas for the proof of the third one we split the integral in the following way:

$$\begin{aligned} \int_{-\infty}^{\infty} |I_k(\chi,\eta)|^{2m} \delta\eta \ll & \int_{|\eta| \le N_k^{-k}} N_k^{2m} d\eta + \int_{N_k^{-k} < |\eta| \le \frac{T}{k\pi M_k^k}} N_k^{2m} M_k^{2m-3mk} |\eta|^{-m} d\eta \\ & + \int_{\frac{T}{k\pi M_k^k} \le |\eta|} M_k^{(1-k)2m} |\eta|^{-2m} d\eta \ll \frac{N_k^{3mk-k}}{M_k^{3mk-2m}}. \end{aligned}$$

We now simplify $I_1(N)$ as in [6]. Set $G_k(a,q,\eta) = \sum_{\chi \bmod q} C_k(\overline{\chi}, a) I_k(\chi, \eta)$ and

$$H_k(a,q,\eta) = C_k(\chi_0, a) I_k(\eta) - \delta_q C_k(\tilde{\chi}\chi_0, a) \tilde{I}_k(\eta) - G_k(a,q,\eta),$$

where $\delta_q = \left\{ \begin{matrix} 1, & \text{if } \tilde{r} | q, \\ 0, & \text{otherwise.} \end{matrix} \right\}$.

For any $\alpha = \frac{a}{q} + \eta \in I(a,q)$, with the help of Lemma 3.1 we get for (3.1) that

$$S_k(\alpha) = \phi^{-1}(q) \left(H_k(a,q,\eta) + O\left(\sum_{\chi \bmod q} (1 + |\eta| N) |C_{\overline{\chi}}(a)| N^{1/k} L^2 T^{-1} \right) \right).$$

Arguing in the same way as in [7], Section 3, we obtain by using Lemmas 3.3 and 3.4 for a sufficiently small δ_1 that

$$I_1(N) = \sum_{q \le P} \phi^{-4}(q) \sum_{a=1}^{q}{}^{*} e\left(-\frac{aN}{q}\right) \int_{-\infty}^{\infty} e(-\eta N) \prod_{k=2}^{5} H_k(a,q,\eta) + O(N^{\mu} P^{-1}). \qquad (3.2)$$

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4 Lemmas for the Treatment of Singular Series and Singular Integral

In this section we follow very closely the arguments in Section 5 in [5]. We define for any characters $\chi_j (\mathrm{mod}\ q), j \in \{2,3,4,5\}$:

$$Z(q,N,\chi_2,\chi_3,\chi_4,\chi_5)(=Z(q,N)) = \sum_{h=1}^{q}{}^{*} e\left(\frac{-hN}{q}\right) \prod_{k=2}^{5} C_k(\chi_k,h), \tag{4.1}$$

and

$$Y(q,N,\chi_2,\chi_3,\chi_4,\chi_5)(=Y(q,N)) = \sum_{h=1}^{q} e\left(\frac{-hN}{q}\right) \prod_{k=2}^{5} C_k(\chi_k,h).$$

Using the definition of the $C_k(\cdots)$ we obtain

$$Y(q,N,\chi_2,\chi_3,\chi_4,\chi_5) = q \sum_{\substack{1\le n_k\le q \\ (n_k,q)=1 \\ n_2^2+\cdots+n_5^5\equiv N (\mathrm{mod}\, q)}} \chi_2(n_2)\chi_3(n_3)\chi_4(n_4)\chi_5(n_5). \tag{4.2}$$

We further set

$$A(q,N) = \phi^{-4}(q) Z(q,N,\chi_0,\chi_0,\chi_0,\chi_0), \tag{4.3}$$

and

$$N(q,N) = \sum_{\substack{1\le n_k\le q \\ (n_k,q)=1 \\ n_2^2+\cdots+n_5^5\equiv N (\mathrm{mod}\, q)}} 1,$$

such that by (4.2)

$$Y(q,N,\chi_0,\chi_0,\chi_0,\chi_0) = qN(q,N). \tag{4.4}$$

The following Lemmas 4.1–4.4 can be proved in the same way as Lemmas 5.1 to 5.4 in [5]. So we omit the proofs.

Lemma 4.1 *$N(q,N), Z(q,N), Y(q,N)$ and $A(q,N)$ are multiplicative functions of q.*

Lemma 4.2 *For any positive integer q we have $\phi^{-4}(q)Z(q,N) \ll q^{-1+\epsilon}$.*

Lemma 4.3 *For $k \in \{2,3,4,5\}$ let $\chi_k (\mathrm{mod}\ p^{\alpha_k})$ be primitive characters and $\alpha = \max(\alpha_2, \cdots, \alpha_5)$. For any $t \ge \alpha$, set $Z(p^t,N) = Z(p^t,N,\chi_2\chi_0,\chi_3\chi_0,\chi_4\chi_0,\chi_5\chi_0)$ and define $Y(p^t,N)$ in the same way.* a) *$Z(p^t,N) = 0$ if $t \ge 1+\max(1,\alpha)$;* b) *$\sum_{v=\alpha}^{\eta} \phi^{-4}(p^v) Z(p^v,N) = \phi^{-4}(p^\eta) Y(p^\eta,N)$, for any $\eta \ge \alpha$.*

Lemma 4.4 a) *$A(p^v,N) = 0$ for $v \ge 2$;* b) *$p^\eta \phi^{-4}(p^\eta) N(p^\eta,N) = p\phi^{-4}(p) N(p,N)$, for $\eta \ge 1$.*

In view of Lemma 4.4 a) we define further

$$s(p,N) = 1 + \sum_{\alpha\ge 1} A(p^\alpha,N) = 1 + A(p,N). \tag{4.5}$$

Remark Prachar has in [3] always taken $A(4,n)$ not necessarily to be equal to zero, whereas our Lemma 4.4 a) shows that this is wrong. Following Prachar's arguments this can also been seen by noting that Hua's lemma (see Prachar Lemma 4) can be sharpened in the case $p = 2, 2 \nmid k$ to $\gamma = r + 1$.

Lemma 4.5 *We have* $\prod_{p\le P} s(p,N) \gg \log^{-960} P$.

Proof We first note that by Lemma 13 in [3] $s(p,n)\neq 0$ for all even n. Then we obtain by the definition of $s(p,n)$ and the estimate

$$|C_k(\chi,a)| \le kp^{1/2}. \tag{4.6}$$

(for $\chi(\bmod\ p)$ and $(a,p)=1$; for the proof see [9], Satz 311) that

$$\prod_{p\le P} s(p,N) \gg \prod_{960<p\le P}\left(1-\frac{960}{p}\right) \gg \log^{-960} P.$$

For the treatment of the singular integrals we need the following lemma:

Lemma 4.6 *For any complex numbers ρ_i with $0 < Re(\rho_j) \le 1, j = 2,\cdots,5$, we have*

$$\int_{-\infty}^{\infty} e(-N\eta)\prod_{k=2}^{5}\left(\int_{M_k}^{N_k} x^{\rho_j-1}e(x^k\eta)dx\right)d\eta = N^{\mu}\frac{1}{5!}\int_{\mathcal{D}}\prod_{k=2}^{5}(Nx_k)^{(\rho_j-1)/k}x_k^{\frac{1-k}{k}}dx_2dx_3dx_4, \tag{4.7}$$

where

$$x_5 = 1-\sum_{k=2}^{4} x_k \tag{4.8}$$

and

$$\mathcal{D} = \{(x_2,\cdots,x_5) : M/N \le x_2,\cdots,x_5 \le 1\}. \tag{4.9}$$

Further there holds

$$\int_{\mathcal{D}}\prod_{k=2}^{5} x_k^{\frac{1-k}{k}}dx_2dx_3dx_4 \gg P^{\frac{-13\xi}{192}}. \tag{4.10}$$

Proof (4.7) is shown in exactly the same way as (5.7) in [5]. For the proof of (4.10) we note that for x_5 (4.9) is equivalent to $0\le \sum_{k=2}^{4} x_k \le 1-P^{\frac{-\xi}{16}}$. We define the region $\mathcal{D}_1$ by

$$\mathcal{D}_1 = \{(x_2,x_3,x_4) : P^{\frac{-\xi}{16}} \le x_2,x_3,x_4 \le \frac{1}{3}(1-P^{\frac{\xi}{16}}),$$

and see that $\mathcal{D}_1$ lies in $\mathcal{D}$. So we get by $x_5^{-4/5}\ge 1$ that

$$\int_{\mathcal{D}}\prod_{k=2}^{5} x_k^{\frac{1-k}{k}}dx_2dx_3dx_4 \ge \int_{\mathcal{D}_1}\prod_{k=2}^{5} x^{\frac{1-k}{k}}dx_2dx_3dx_4 \gg P^{\frac{-\xi}{16}(\frac{1}{2}+\frac{1}{3}+\frac{1}{4})} = P^{\frac{-13\xi}{192}}.$$

5 The Singular Series and the Singular Product

Lemma 5.1 *For any R with $1\le R\le P$ and any integer r we have*

$$\sum_{N\le x}\Big|\prod_{\substack{p\le R\\(p,r)=1}} s(p,N) - \sum_{\substack{q\le P\\(q,r)=1}} A(q,N)\Big| \ll xR^{-\frac{1}{3}+\epsilon}, \tag{5.1}$$

which implies that for all but $\ll xR^{-\frac{1}{6}+\epsilon}$ *even integers* N *with* $1 \le N \le x$ *there holds*

$$\prod_{\substack{p\le P\\(p,r)=1}} s(p,N) = \sum_{\substack{q\le P\\(q,r)=1}} A(q,N) + O(R^{-\frac{1}{6}}). \tag{5.2}$$

Proof Denoting the left-hand side in (5.1) by J we first have

$$J \le \Big|\sum_{\substack{R<q<V\\q\in\mathcal{D}_T^*}} A(q,N)\Big| + \Big|\sum_{\substack{q\ge V\\q\in\mathcal{D}_T^*}} A(q,N)\Big| =: T_1(N,R)+T_2(N,R), \tag{5.3}$$

with $V = \exp\left(\frac{\log P\log x}{\log\log x}\right)$, $\mathcal{D}_r = \{q : \mu(q)\ne 0, (q,r)=1, p|q \Longrightarrow p\le P\}$, where the condition $\mu(q)\ne 0$ is due to Lemma 4.4 a). Lemma 4.3 in [10] says that for $(m,p)=1$, $\sum_{a=1}^{p} e\left(\frac{a^k m}{p}\right) = \sum_{\chi\in\mathcal{A}_p^k}\overline{\chi(m)}\tau(\chi)$, where $\mathcal{A}_p^k = \{\chi(\text{mod } p) : \chi^k=\chi_0, \chi\ne\chi_0\}$ and

$$\text{card}\mathcal{A}_p^k = (k,p-1)-1. \tag{5.4}$$

So we have for $(p,N)=1$

$$\begin{aligned}
(p-1)^4A(p,N) &= \sum_{h=1}^{p-1} e\left(\frac{-hN}{p}\right)\prod_{k=2}^{5}\left(\sum_{\chi\in\mathcal{A}_p^k}\overline{\chi(h)}\tau(\chi)-1\right)\\
&= \sum_{l=1}^{4}(-1)^l \sum_{\substack{k_1,\cdots,k_l\in\{2,\cdots,5\}\\k_i<k_j(i<j)}}\ \sum_{\chi_1\in\mathcal{A}_p^{k_1}}\tau(\chi_1)\cdots\sum_{\chi_l\in\mathcal{A}_p^{k_l}}\tau(\chi_l)\sum_{h=1}^{p-1}\overline{\chi_1\cdots\chi_l}(h)e\left(\frac{-hN}{p}\right)+e\left(\frac{-hN}{p}\right)\\
&= \sum_{l=1}^{4}(-1)^l \sum_{\substack{k_1,\cdots,k_l\in\{2,\cdots,5\}\\k_i<k_j(ij)}}\ \sum_{\chi_1\in\mathcal{A}_p^{k_1}}\tau(\chi_1)\cdots\sum_{\chi_l\in\mathcal{A}_p^{k_l}}\tau(\chi_l)\tau(\overline{\chi_1\cdots\chi_l})\chi_1\cdots\chi_l(-N)+e\left(\frac{-hN}{p}\right)\\
&= \sum_{\chi\in\mathcal{B}_p,\chi\ne\chi_0}\chi(-N)f(\chi)+C(p),
\end{aligned} \tag{5.5}$$

where by $|\tau(\chi)|\le p^{1/2}$ and (5.4) there holds

$$\text{card}\mathcal{B}_p \le 4\times 5! = 480,\quad |f(\chi)|\le 480p^{5/2},\quad |C(p)|\le 480p^2+1. \tag{5.6}$$

Taking note of Lemmas 4.1 and 4.4 a) we use (5.5) to define for $q\in\mathcal{D}_r$

$$A(q,N) = \sum_{m|q} A_1(m,N)A_2(q/m,N), \tag{5.7}$$

where

$$A_1(p,N) = \begin{cases}\frac{1}{(p-1)^4}\sum_{\chi\in\mathcal{B}_p,\chi\ne\chi_0}\chi(-N)f(\chi), & \text{if } (p,N)=1,\\ 0, & \text{if } p|N.\end{cases}$$

$$A_2(p,N) = \begin{cases}\frac{C(p)}{(p-1)^4}, & \text{if } (p,N)=1,\\ A(p,N), & \text{if } p|N.\end{cases}\qquad A_i(q,N) = \prod_{p|q}A_i(p,N), i\in\{1,2\},$$

and an empty product is equal to 1. By (5.3) and (5.7) we have

$$\begin{aligned} T_1(N,R) \le & \sum_{\substack{R^{1/3}<m<V \\ m\in\mathcal{D}_r}} |A_2(m,N)| \sum_{\substack{R/m<d<V/m,(d,m)=1 \\ d\in\mathcal{D}_r}} |A_1(d,N)| \\ & + \sum_{\substack{m\le R^{1/3} \\ m\in\mathcal{D}_r}} |A_2(m,N)| \Big| \sum_{\substack{R/m<d<V/m,(d,m)=1 \\ d\in\mathcal{D}_r}} A_1(d,N) \Big| \\ := & F_1(N,R)+F_2(N,R). \end{aligned} \tag{5.8}$$

We deduce from (4.6) that $|A(p,N)| \le \frac{120p^2}{(p-1)^3}$, and so we get together with (5.6) that

$$\begin{aligned} F_1(N,R) & \le R^{-1/3} \sum_{\substack{m<V \\ m\in\mathcal{D}_r}} m|A_2(m,N)| \sum_{\substack{m<V \\ d\in\mathcal{D}_r}} |A_1(d,N)| \\ & \le R^{-1/3} \prod_{p\le P} (1+p|A_2(p,N)|) \prod_{\substack{p\le P \\ (p,n)=1}} (1+A_1(p,N)|) \\ & \le R^{-1/3} \prod_{\substack{p\le P \\ (p,N)=1}} \left(1+\frac{c}{p}\right) c^{\omega(N)} \prod_{p\le P} \left(1+\frac{c}{p^{3/2}}\right) \\ & \ll R^{-1/3} c^{\omega(N)} (\log P)^c. \end{aligned} \tag{5.9}$$

Now we get by the definition of $A_1(d,N)$:

$$\sum_{\substack{R/m<d<V/m,(d,m)=1 \\ d\in\mathcal{D}_r}} A_1(d,N) = \sum_{\substack{R/m<d<V/m,(d,m)=1 \\ d\in\mathcal{D}_r}} \sum_{\chi \bmod d} {}^{*} g(\chi)\chi(-N)$$

where

$$|g(\chi)| \le \begin{cases} \prod_{p|d} \frac{480p^{5/2}}{(p-1)^4}, & \text{if } \chi = \prod_{p|q} \chi_p \text{ with } \chi_p \in \mathcal{B}(p) \quad \forall p|d, \\ 0, & \text{otherwise.} \end{cases}$$

By (5.6) we have for any positive number a and any $d \in \mathcal{D}_r$:

$$\sum_{\chi \bmod d} {}^{*} |g(\chi)|^a \le (480)^{\omega(d)} \prod_{p|d} \frac{(480\times 16)^a}{p^{3a/2}} \le \frac{(c^a)^{\omega(d)}}{d^{3a/2}}. \tag{5.10}$$

Further we write

$$[R/m, V/m] \subseteq \bigcup_{j=1}^{H} [Q_{j-1}, Q_j], \tag{5.11}$$

with $Q_0 = R/m, Q_j = x^{j/2}, j=1,\cdots,H; H \le \frac{\log P}{\log\log x}$. Then for a fixed j we get by (5.10), (5.11) and Lemma 6.5 in [5]:

$$\sum_{N\le x}\Big|\sum_{\substack{Q_{j-1}<d\le Q_j\\ d\in\mathcal{D}_r}}\ \sum_{\chi \bmod d}{}^{*} g(\chi)\chi(-N)\Big|$$

$$\ll (x^{1/2}+Q_j^{1/j})x^{1/2}(\log(x^j e))^{(j^2-1)/2j}\left(\sum_{Q_{j-1}<d\le Q_j}\ \sum_{\chi \bmod d}{}^{*}|g(\chi)|^{2j/(2j-1)}\right)^{(2j-1)/2j}$$

$$\ll x(\log(x^j e))^{(j^2-1)/2j}\left(Q_{j-1}^{\frac{-1}{2j-1}}\sum_{Q_{j-1}<d\le Q_j}\frac{c^{\omega(d)}}{d^{3/2}}\right)^{(2j-1)/2j} \ll x(\log(x^j e))^{(j^2-1)/2j}Q_{j-1}^{-1/2j} \tag{5.12}$$

From (5.11) and (5.12) we get

$$\sum_{N\le x}\Big|\sum_{\substack{R/m<d<V/m,(d,m)=1\\ d\in\mathcal{D}}} A_1(d,N)\Big| \ll xQ_0^{-1/2}+x^{7/8}\sum_{j=2}^{L}(\log x^{j+1})^{(j^2-1)/2j}. \tag{5.13}$$

For the sum in (5.13) we obtain for a sufficiently small δ_1

$$\sum_{j=2}^{H}(\cdots)\le\sum_{j=2}^{H}((j+1)\log x)^{j/2}\le 2\frac{\log P}{\log\log x}\left(3\frac{\log P}{\log\log x}\log x\right)^{\frac{\log P}{\log\log x}} \ll P^3. \tag{5.14}$$

So we find by (5.13), (5.14), the definition of Q_0 and $m\le R^{1/3}$ for a sufficiently small δ that

$$\sum_{N\le x}\Big|\sum_{\substack{R/m<d<V/m(d,m)=1\\ d\in\mathcal{D}}} A_1(d,N)\Big| \ll x(R^{-\frac{1}{2}\frac{2}{3}}+P^3x^{-1/8})\ll xR^{-1/3}. \tag{5.15}$$

For the final estimate of $F_2(N,r)$ we note

$$\sum_{\substack{m\le R^{1/3}\\ m\in\mathcal{D}}}|A_2(m,N)|\le\prod_{p\le P}\left(1+\frac{480p^2}{(p-1)^4}\right)\prod_{p\le P}\left(1+\frac{c}{p}\right)\ll(\log P)^c. \tag{5.16}$$

For $T_2(N,R)$ we get with $v=\dfrac{\log\log x}{2\log P}$:

$$T_2(N,R)\le\sum_{q\in\mathcal{D}_r}\left(\frac{q}{V}\right)^v|A(q,N)|\le V^{-v}\prod_{2<p\le P}(1+p^v|A(p,N)|).$$

By $V^{-v}=x^{-1/2}$ and $p^v\le(\log x)^{1/2}$, we obtain for a sufficiently small δ

$$T_2(N,R)\le x^{-1/2}\prod_{p\le P}\left(1+\frac{c(\log x)^{1/2}}{p}\right)\ll x^{-1/2}(\log P)^{c(\log x)^{1/2}}\ll x^{-1/3}. \tag{5.17}$$

So we obtain from (5.3), (5.8), (5.9), (5.15), (5.16) and (5.17): $J\ll xR^{-1/3+\epsilon}$.

Lemma 5.2 *Let $\chi_j(\mathrm{mod}\, r_j), j\in\{1,2,3\}$ denote primitive characters and r be the least common multiple of the r_j. We define in the following $Z(q,N)$ and $Y(r,N)$ separately with the characters $\chi_2\chi_0,\chi_3\chi_0,\chi_4\chi_0,\chi_5\chi_0$.*

a) $\displaystyle\sum_{q\le P,r|q}\phi^{-4}(q)Z(q,N)\ll_\epsilon r^{-1}P^\epsilon$;

b) $$\sum_{N\le x}\left|\sum_{q\le P,r|q}\phi^{-4}(q)Z(q,N)-\phi^{-4}(r)Y(r,N)\prod_{p\le P,(p,r)=1}s(p,N)\right|\ll xP^{-1/3+\epsilon};$$

here b) *implies that*

$$\sum_{q\le P,r|q}\phi^{-4}(q)Z(q,N)=\phi^{-4}(r)Y(r,N)\prod_{p\le P,(p,r)=1}s(p,N)+O(P^{-1/16}), \tag{5.18}$$

for all but $\ll xP^{-1/6+\epsilon}$ *even integers* $N\ll x$.

Proof a) We find by Lemmas 4.1, 4.2, 4.3 a) and (4.3):

$$\begin{aligned}\left|\sum_{q\le P,r|q}\phi^{-4}(q)Z(q,N)\right| &\le|\phi^{-4}(r)Z(r,N)|\sum_{q\le P/r,(q,r)=1}|A(q,N)|\\ &\ll r^{-1+\epsilon}\sum_{q\le P/r}|A(q,N)|\ll r^{-1}P^{\epsilon}\end{aligned}$$

b) We see by Lemmas 4.1,4.2 and 4.3 that

$$\begin{aligned}&\sum_{N\le x}\left|\sum_{q\le P,r|q}\phi^{-4}(q)Z(q,N)-\phi^{-4}(r)Y(r,N)\prod_{p\le P,(p,r)=1}s(p,N)\right|\\ &=\sum_{N\le x}\left|\phi^{-4}(r)Z(r,N)\sum_{q\le P/r,(q,r)=1}A(q,N)-\phi^{-4}(r)Z(r,N)\prod_{p\le P,(p,r)=1}s(p,N)\right|\\ &\ll r^{-1+\epsilon}\sum_{n\le x}\left|\sum_{q\le P/r,(q,r)=1}A(q,N)-\prod_{p\le P,(p,r)=1}s(p,N)\right|.\end{aligned}$$

If $r\le P$, we derive the lemma from (5.1). In the other case we first note that by (4.6) $|s(p)|\le 1+960/p$. Using this and arguing again as in the first case we get

$$\begin{aligned}\left|\phi^{-4}(r)Y(r,N)\prod_{p\le P,(p,r)=1}s(p,N)\right| &\le\phi^{-4}(r)Z(r,N)\prod_{p\le P,(p,r)=1}s(p,N)\\ &\ll r^{-1+\epsilon}\prod_{p\le P,(p,r)=1}\left(1+\frac{960}{p}\right)\ll P^{-1+\epsilon}.\end{aligned} \tag{5.19}$$

Then the lemma follows trivially from (5.19) and part a).

6 The Major Arcs

The treatment of the major arcs is very similar to the procedure in [5]. So we will only explain briefly how the proofs are derived; for more details refer to [5]. We know from the definition of $H_k(a,q,\eta)$ that $\prod_{k=2}^{5}H_k(a,q,\eta)$ is a sum of 3^4 terms which can be divided into three groups:

T_1 : the term $\prod_{k=2}^{5}C_k(\chi_0,a)I_k(\eta)$,

T_2 : the 65 terms each of which has at least one $G(a,q,\eta)$ as factor,

T_3 : the remaining 15 terms.

We further write for $i=1,2,3$,

$$M_i=\sum_{q\le P}\phi^{-5}(q)\sum_{a=1}^{q}{}^{*}e\left(\frac{-Na}{q}\right)\int_{-\infty}^{\infty}e(-N\eta)\{\text{sum of the terms in } T_i\}d\eta,$$

from which we deduce by (3.2) that

$$I_1(N) = M_1 + M_2 + M_3 + O(N^{\mu}P^{-1}). \tag{6.1}$$

We set

$$\mathcal{P}_0 = \frac{N^{\mu}}{5!}\int_{\mathcal{D}}\prod_{k=2}^{5} x_k^{\frac{1-k}{k}}\,dx_2dx_3dx_4, \tag{6.2}$$

and we see by (4.9):

$$\mathcal{P}_0 \ll N^{\mu}P^{\xi/16}. \tag{6.3}$$

We further easily derive from (4.5) that $s(p,N) = \frac{pN(p,N)}{(p-1)^4}$, and so we get by Lemmas 4.1 and 4.4 b):

$$\prod_{p|\tilde{r}} s(p,N) = \tilde{r}\phi^{-4}(\tilde{r})N(\tilde{r},N). \tag{6.4}$$

We will now give some lemmas for the contribution of the M_i to $I_1(N)$. We first have

Lemma 6.1

$$M_1 = \mathcal{P}_0\prod_{p\le P} s(p,N) + O(N^{\mu}P^{\frac{-1}{6}+\frac{\xi}{16}}), \tag{6.5}$$

for all but $\ll xP^{-1/6+\epsilon}$ *even integers* $N \le x$.

Proof This is proved in exactly the same way as Lemma 7.1 in [5] by using Lemma 4.6, (5.2), (6.2) and (6.3).

Making use of Lemmas 4.6, 5.2 a) and (6.3) for part a), and Lemma 4.6), (4.4), (5.18), (6.3) and (6.4) for part b), we get similarly to the proofs of Lemmas 7.2 and 7.3 in [5]:

Lemma 6.2 a) $M_3 \ll N^{\mu}\tilde{r}^{-1}P^{\frac{\xi}{16}+\epsilon}$; b) $M_1 + M_3 \ge \Omega^4\mathcal{P}_0\prod_{p\le P} s(p,N) + O(N^{\mu}P^{\frac{-1}{6}+\frac{\xi}{16}})$,

for all but $\ll xP^{-1/6+\epsilon}$ *even integers* $N \le x$.

For M_2, by using Lemmas 2.1, 4.6, 5.2 b) and (6.3) we obtain in the same way as Lemma 7.4 in [5]:

Lemma 6.3 *There exists an absolute constant* c_4 *such that*

$$M_2 \ll \Omega^4\exp(-c_4/\delta)\mathcal{P}_0\prod_{p\le P} s(p,N) + O(N^{\mu}P^{\frac{-1}{6}+\frac{\xi}{16}}),$$

for all but $\ll xP^{-1/6+\epsilon}$ *even integers* $N \le x$.

From these three lemmas we derive for a sufficiently small δ_1 a lower bound for the contribution of the major arcs by distinguishing three cases:

a) $\tilde{r}$ does not exist. By applying Lemmas 4.5, 6.1, 6.3 and (4.10) to (6.1) we obtain

$$I_1(N) \ge \frac{1}{2}\mathcal{P}_0\prod_{p\le P} s(p,N) + O(N^{\mu}P^{\frac{-1}{6}+\frac{\xi}{16}}) \gg N^{\mu}P^{-\xi/8}, \tag{6.6}$$

for all but $\ll xP^{-1/6+\epsilon}$ even integers $N \le x$.

b) $\tilde{r}$ exists and $\tilde{r} \ge P^{\xi/2}$. We derive from Lemmas 4.5, 6.1, 6.2 a), 6.3 and (4.10) that

$$I_1(N) \ge \frac{1}{2}\mathcal{P}_0\prod_{p\le P} s(p,N) + O(N^{\mu}P^{-\xi/4}) \gg N^{\mu}P^{-\xi/8}, \tag{6.7}$$

for all but $\ll xP^{-1/6+\epsilon}$ even integers $N \leq x$.

c) $\tilde{r}$ exists and $\tilde{r} < P^{\xi/2}$. From Lemmas 6.2 b) and 6.3 we get

$$I_1(N) \geq \frac{1}{2}\omega^4 \mathcal{P}_0 \prod_{p \leq P} s(p, N) + O(N^\mu P^{\frac{-1}{6}+\frac{\xi}{16}})$$

for all but $\ll xP^{-1/6+\epsilon}$ even integers $N \leq x$.

From (2.4) and (2.5) we get $\Omega = (1-\tilde{\beta})\log T \geq (c_3 \log T)(\tilde{r}^{1/2}\log\tilde{r})^{-1} \gg P^{-\xi/4}\log^{-1} T$, from which we deduce by Lemma 4.5 and (4.10) that

$$I_1(N) \gg N^\mu P^{-9\xi/8}, \tag{6.8}$$

for all but $\ll xP^{-1/6+\epsilon}$ even integers $N \leq x$.

7 The Minor Arcs

We quote Lemma 3 in [3], which satates together with (2.1) that $\int_0^1 |S_2(\alpha)S_3(\alpha)S_4(\alpha)|^2 \ll N^{2\mu+\frac{3}{5}}$; and we know from Theorem 1 in [11] that $S_5(\alpha) \ll N^{1/5+\epsilon}P^{-4\xi}$, for $\alpha \in m$. Combining these two results and using Bessel's inequality we obtain

$$\begin{aligned}\sum_{x/2 \leq N < x} |I_2(N)|^2 &\leq \int_m |S_2(\alpha)S_3(\alpha)S_4(\alpha)S_5(\alpha)|^2 \\ &\leq \max_{\alpha\in m}|S_5(\alpha)|^2 \int_m |S_2(\alpha)S_3(\alpha)S_4(\alpha)|^2 \ll x^{2\mu+1}P^{-7\xi},\end{aligned}$$

from which we derive that

$$I_2(N) \ll N^\mu P^{-3\xi} \tag{7.1}$$

for all but $\ll xP^{-\xi}$ even integers $x/2 \leq N < x$. We can now derive the theorem from (2.6), (6.6), (6.7), (6.8) and (7.1), by splitting the interval $[1, x]$ into intervals of the type $[t, 2t]$.

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