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# Hua's theorem for five almost equal prime squares

By

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**Abstract.** Let $p_i$, $1 \leqq i \leqq 5$, be prime numbers. It is proved that every sufficiently large integer $N$ that satisfies $N \equiv 5 (mod\, 24)$ can be written as $N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2$, where $|\sqrt{\frac{N}{5}} - p_i| \leqq N^{\frac{1}{2} - \frac{9}{280} + \epsilon}$.

**1. Introduction.** One of the important contributions of Hua to additive number theory is the theorem on the sum of five prime squares. Hua proved [5] that every sufficiently large integer $n$ satisfying $n \equiv 5 (mod\, 24)$ is equal to the sum of five prime squares. Assuming the Generalized Riemann Hypothesis, Liu and Zhan [8] considered a generalization of this problem for primes in short intervals. They showed the following result:

*Assume the General Riemann Hypothesis. Then any sufficiently large integer n satisfying* $n \equiv 5 (mod\, 24)$ *can be written as*

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2,$$

$$\text{(1.1)} \qquad \textit{where} \left| p_i - \sqrt{\frac{n}{5}} \right| \leqq y, \ i = 1, 2, 3, 4, 5$$

*for*

$$y = n^{\frac{9}{20} + \epsilon}.$$

In a series of papers [1], [2], [9] the same problem was investigated without any assumption on the distribution of the zeros of the Dirichlet $L$-functions. So far, the best result was obtained in [2]:

*Set*

$$y = n^{\frac{1}{2} - \frac{19}{850} + \epsilon}.$$

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*Any sufficiently large positive integer n satisfying* $n \equiv 5 (mod\, 24)$ *can be written as*

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2, \tag{1.2}$$
$$\text{where } \left| p_i - \sqrt{\frac{n}{5}} \right| \leqq y, \; i = 1, 2, 3, 4, 5.$$

In this paper, we improve on this result by applying a method introduced by Liu in [7] and a new estimate for Dirichlet polynomials introduced by Choi and Kumchev [6]. The novelty of Liu's method lies in the evaluation of the contribution of the partial singular series containing characters sums that naturally arise in the problem. Previously in [2], for each module $r$ of a primitive characters one saves a factor $r^{-3/2+\epsilon}$ when evaluating the partial singular series. The saving $r^{-3/2+\epsilon}$ is divided among the contribution of the five different sums over primes involved in the problem. In contrast, by applying the method introduced in [7], here we use an iterative method which allows us to save the full factor $r^{-3/2+\epsilon}$ for all five partial singular series. Applying the new estimate for Dirichlet polynomials in [6], this allows us to improve upon the result in [2] and we prove the following:

**Theorem 1.** *Any sufficiently large positive integer n satisfying* $n \equiv 5 (\mathrm{mod}\, 24)$ *can be written as*

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2, \tag{1.3}$$
$$\text{where } \left| p_i - \sqrt{\frac{n}{5}} \right| \leqq y, \; i = 1, 2, 3, 4, 5$$

*for*

$$y = n^{\frac{1}{2} - \frac{9}{280} + \epsilon}.$$

We note that the application of only the method of Liu [7] without applying the new estimate for Dirichlet polyomials [6] would have established Theorem 1 with $y = n^{\frac{1}{2} - \frac{1}{35} + \epsilon}$, which is also better than the previous best bound of [2].

After this work was finished, the author learned of the yet unpublished work in [10] and [11] in which the value of $y$ is further reduced to $y = n^{\frac{1}{2} - \frac{1}{20} + \epsilon}$ by introducing a new technique on the *major* arcs.

**2. Preliminaries and Outline of the proof.** $(a, b)$ and $[a, b]$ denote the greatest common divisor and the smallest common multiple of two integers $a$ and $b$ respectively. Let

$$L = \log n, \quad e(x) = e^{2\pi i x}, \; N_1 = \sqrt{\frac{n}{5}} - y, \; N_2 = \sqrt{\frac{n}{5}} + y,$$
$$\sum_{a=1}^{q\;*} := \sum_{\substack{a=1 \\ (a,q)=1}}^{q}, \quad \sideset{}{^*}\sum_{\chi \bmod q} := \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}},$$

$$S(\alpha)=\sum_{N_1\leqq m\leqq N_2}\Lambda(m)e(m^2\alpha),$$
$$R(n)=\sum_{\substack{n=m_1^2+m_2^2+m_3^2+m_4^2+m_5^2\\ N_1<m_i\leqq N_2}}\Lambda(m_1)\Lambda(m_2)\Lambda(m_3)\Lambda(m_4)\Lambda(m_5).$$

Define for a character $\chi \mod q$

$$C(a,\chi)=\sum_{h=1}^{q}\chi(h)e\left(\frac{a}{q}h^2\right),\quad C(a,q):=C(a,\chi_0).$$

Let $c$ and $\epsilon,\epsilon_1,..>0$ denote constants that may take different values on different occasions. We shall write $x^{\epsilon}L^{c}\ll x^{\epsilon}$, $x^{\epsilon_1}x^{\epsilon_1}\ll x^{\epsilon_1}$. Set

(2.1) $$P=n^{2+\epsilon_1}y^{-4},\ \ Q=y^{7}n^{-\frac{5}{2}-2\epsilon_1}.$$

We define the major arcs $M$ and the minor arcs $m$ by

$$M=\bigcup_{q\leqq P}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q}\left[\frac{a}{q}-\frac{1}{Qq},\ \frac{a}{q}+\frac{1}{Qq}\right],$$
$$m=\left[-\frac{1}{Q},\ 1-\frac{1}{Q}\right]\setminus M.$$

We have

$$R(n)=\int_M S^5(\alpha)e(-n\alpha)\,d\alpha+\int_m S^5(\alpha)e(-n\alpha)\,d\alpha$$
(2.2) $$=:R_1(n)+R_2(n).$$

We will prove that $R(n)>0$ which proves Theorem 1.

For the treatment of the minor arcs we quote the following lemma due to Harman, [4].

**Lemma 2.1.** *Suppose $\epsilon>0$ is given and*

$$\left|\frac{a}{q}-\alpha\right|<q^{-1}\qquad with\ (a,q)=1.$$

*Then*

$$\sum_{x\leqq n\leqq x+y}\Lambda(n)\,e(n^2\alpha)\ll y^{1+\epsilon}\left(\frac{1}{q}+\frac{x^{\frac{1}{2}}}{y}+\frac{x^{\frac{4}{3}}}{y^2}+\frac{qx}{y^3}\right)^{\frac{1}{4}}$$

*holds for $1\leqq q\leqq xy$.*

Applying this to $S(\alpha)$ we find that

$$\begin{aligned}\max_{\alpha\in m}|S(\alpha)| &\ll y^{1+\epsilon}\left(P^{-1/4}+\frac{n^{\frac{1}{16}}}{y^{\frac{1}{4}}}+\frac{n^{\frac{1}{6}}}{y^{\frac{1}{2}}}+\frac{Q^{\frac{1}{4}}n^{\frac{1}{8}}}{y^{\frac{3}{4}}}\right)\\ &\ll y^2 n^{-\frac{1}{2}-\epsilon_1/8} \end{aligned}\tag{2.3}$$

by choosing $\epsilon_1 \geqq 8\epsilon$. Using (2.3) and the so called Hua's Lemma [5], we estimate the contribution of the minor arcs as

$$R_2(n) \leqq \sup_{\alpha\in m}|S(\alpha)|\int_0^1 |S(\alpha)|^4\,d\alpha \ \ll\ y^4 n^{-\frac{1}{2}}L^{-B}. \tag{2.4}$$

for any $B > 0$.

In the following sections we shall first show that for any $B > 0$

$$R_1(n) = \frac{1}{32}P_0\sum_{q\leqq P}\frac{Y(q)}{\phi^5(q)} + O(y^4 n^{-1/2}L^{-B}), \tag{2.5}$$

where

$$y^4 n^{-1/2} \ll P_0 = \sum_{\substack{m_1+m_2+m_3+m_4+m_5=n\\ N_1^2<m_i\leqq N_2^2}} \frac{1}{\sqrt{m_1m_2m_3m_4m_5}} \ll y^4 n^{-1/2}, \tag{2.6}$$

and

$$\begin{aligned} Y(q) &:= Z(q,\chi_0,\chi_0,\chi_0,\chi_0,\chi_0),\\ Z(q,\chi_1,\chi_2,\chi_3,\chi_4,\chi_5) &:= \sum_{a=1}^{q}{}^{*}\\ &\quad C(a,\chi_1)C(a,\chi_2)C(a,\chi_3)C(a,\chi_4)C(a,\chi_5)e\left(-\frac{a}{q}n\right),\\ A(q) := \frac{Y(q)}{\phi^5(q)}, &\quad s(p) := \begin{cases} 1+A(p), & p>2,\\ 1+A(2)+A(4)+A(8), & p=2.\end{cases} \end{aligned}$$

Finally we will derive

$$R_1(n) = \frac{1}{32}P_0\prod_{p\geqq 1}s(p) + O(y^4 n^{-1/2}L^{-B}), \tag{2.7}$$

where $\prod_{p\geqq 1} s(p) > c$, from (2.5). Theorem 1 follows from (2.2), (2.4), (2.6) and (2.7).

**3. Main lemmas.** We define the following quantities which we will need for the proof of Theorem 1:

$$S(\lambda,\chi)=\sum_{N_1<m\leqq N_2}\Lambda(m)\chi(m)e(m^2\lambda),\ \ T(\lambda)=\sum_{N_1<m\leqq N_2}e(m^2\lambda),$$

$$W(\lambda,\chi)=S(\lambda,\chi)-E_0(\chi)T(\lambda),\ \ E_0(\chi)=\begin{cases}1, & \text{if } \chi=\chi_0,\\ 0, & \text{otherwise.}\end{cases}$$

$$J(g)=\sum_{r\leqq P}[g,r]^{-3/2+\epsilon}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,r)}\max_{|\lambda|\leqq 1/rQ}|W(\lambda,\chi)|,$$

$$K(g)=\sum_{r\leqq P}[g,r]^{-3/2+\epsilon}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,r)}\left(\int_{-1/rQ}^{1/rQ}|W(\lambda,\chi)|^2d\lambda\right)^{1/2}.$$

The proof of Theorem 1 will make use of the following lemmas:

**Lemma 3.1.** *If $P\leqq n^{9/70-\epsilon}$ and $g>1$, then*

$$K(g)\ll g^{-3/2+\epsilon}y^{1/2}n^{-1/4}L^c.$$

**Lemma 3.2.** *If $P\leqq n^{3/20-\epsilon}$ and $g>1$, then*

$$J(g)\ll g^{-3/2+\epsilon}yL^c.$$

**Lemma 3.3.** *If $P\leqq n^{3/20-\epsilon}$ then*

$$J(1)\ll yL^{-A},$$

*for any $A>0$.*

**3.1 Proof of Lemma 3.1.** In order to prove the lemma we show that

$$\sum_{r\sim R}[g,r]^{-3/2+\epsilon}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,r)}\left(\int_{-1/rQ}^{1/rQ}|W(\lambda,\chi)|^2d\lambda\right)^{1/2}$$

$$\ll g^{-3/2+\epsilon}y^{1/2}n^{-1/4}L^c \tag{3.1}$$

for $R\leqq P$ and $r\sim R$ denotes $R/2<r\leqq R$. Applying Lemma 1, [3], we see

$$\int_{-1/rQ}^{1/rQ}|W(\lambda,\chi)|^2\,d\lambda$$

$$\ll (QR)^{-2}\int_{N_1^2-QR}^{N_2^2}\left|\sum_{X<m^2\leqq X+Y}\Lambda(m)\chi(m)-E_0(\chi)\sum_{X<m^2\leqq X+Y}1\right|^2dt. \tag{3.2}$$

where $X = \max(t, N_1^2)$, $X+Y = \min(t+Qr, N_2^2)$. Arguing as in [6], we find by applying Perron's summation formula that the inner sum of (3.2) can be written as

$$S := \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s,\chi) \frac{(X+Y)^{\frac{s}{2}} - X^{\frac{s}{2}}}{s} ds + O(T^{-1} n L^2), \tag{3.3}$$

for $T = n$ and $0 < b < L^{-1}$, with

$$F(s,\chi) = \sum_{X^{1/2} \leqq m \leqq (X+Y)^{1/2}} (\Lambda(m)\chi(m) - E_0(\chi)) m^{-s}. \tag{3.4}$$

Using trivial estimates, we see that for $0 < b < L^{-1}$

$$\frac{(X+Y)^{\frac{s}{2}} - X^{\frac{s}{2}}}{s} \ll \min(T_0^{-1}, (|t|+1)^{-1})$$

for $T_0 = n(QR)^{-1}$. Thus, for $b \downarrow 0$, $S$ is bounded by

$$S \ll \int_{-T}^{T} |F(it,\chi)| \frac{dt}{T_0 + |t|} + L^2$$

$$\ll L \max_{T_0 \leqq T_1 \leqq T} \frac{1}{T_1} \int_{-T_1}^{T_1} |F(it,\chi)| \, dt + L^2. \tag{3.5}$$

Thus, we see from (3.2) and (3.5) that the left-hand side of (3.1) is bounded above by

$$\ll L \max_{T_0 \leqq T_1 \leqq T} (QR)^{-1} T_1^{-1} y^{1/2} n^{1/4} \sum_{r \sim R} [g,r]^{-3/2+\epsilon} \sum_{\chi} \int_{-T_1}^{T_1} |F(it,\chi)| \, dt$$

$$+ L^3 (QR)^{-1} y^{1/2} n^{1/4} g^{-3/2} R^2. \tag{3.6}$$

In view of (3.1), we see that the second term in (3.6) is permissible for $P \leqq n^{9/70-\epsilon}$. We note that $[g,r](g,r) = gr$. Thus, the first term of (3.6) is

$$\ll \max_{T_0 \leqq T_1 \leqq T} g^{-3/2+\epsilon} L (QR)^{-1} T_1^{-1} y^{1/2} n^{1/4} \sum_{\substack{d \leqq R \\ d | g}} \left(\frac{R}{d}\right)^{-3/2+\epsilon}$$

$$\cdot \sum_{\substack{r \sim R \\ d | r}} \sideset{}{^*}\sum_{\chi} \int_{-T_1}^{T_1} |F(it,\chi)| \, dt. \tag{3.7}$$

In order to estimate the expression (3.7), we apply Theorem 1.1 from [6]:

**Lemma 3.4.** *For $F(s,\chi)$ defined as in (3.4), there is*

$$\sum_{\substack{r\sim R\\ m|r}}\sideset{}{^*}\sum_{\chi}\int_{-T_1}^{T_1}|F(it,\chi)|\ll\left(n^{1/2}+\frac{R^2T_1}{m}n^{11/40}\right).$$

Applying the lemma and the estimate of divisor function $\sum_{d|g}1\ll g^{\epsilon}$ to (3.7) yields

$$\begin{aligned}&\ll\max_{T_0\leqq T_1\leqq T}\max_{\substack{d|g\\ d\leqq R}}g^{-3/2+\epsilon}L(QR)^{-1}T_1^{-1}y^{1/2}n^{1/4}\\ &\quad\times\left(\frac{R}{d}\right)^{-3/2+\epsilon}\left(n^{1/2}+\frac{R^2T_1}{d}n^{11/40}\right)\\ &\ll g^{-3/2+\epsilon}y^{1/2}n^{-1/4}L^c,\end{aligned}$$

for $P\leqq n^{9/70-\epsilon}$, which is obtained by noting that $Q=n^{1-\epsilon}/P^{7/4}$.

**3.2 Proof of Lemma 3.2.** To prove the lemma it is enough to show that

$$\max_{R\leqq P/2}\max_{0\leqq\Delta\leqq\frac{1}{RQ}}\max_{\Delta\leqq|\lambda|\leqq2\Delta}\sum_{r\sim R}[g,r]^{-3/2+\epsilon}\sideset{}{^*}\sum_{\chi}|W(\lambda,\chi)|\ll g^{-3/2+\epsilon}yL^C. \tag{3.8}$$

We recall the definitions (3.4) and (3.3) from the previous section and in the definition (3.4) change $X$ to $N_1^2$. Then we follow the argument in [6] and see by partial summation that

$$W(\lambda,\chi)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}F(s,\chi)V(s,\lambda)ds+O(1), \tag{3.9}$$

where $0<b\leqq L^{-1}$ and $T=n^{10}$.

$$V(s,\lambda)=\int_{N_1}^{N_2}w^{s-1}e(\lambda w^2)dw.$$

Using a trivial estimates and well-known estimates for exponential integrals as given in [14, Chapter 4], we see for $\Delta\leqq|\lambda|\leqq2\Delta$ :

$$V(\sigma+it,\lambda)\ll n^{\sigma/2}\min\{yn^{-1/2},\sqrt{|t|+1},\max_{N_1\leqq w\leqq2N_1}|t+4\pi\lambda e^2|^{-1}\}. \tag{3.10}$$

We set $T_0=ny^{-2}$, combine (3.9) and (3.10) and let $b\downarrow0$, such that

$$W(\lambda,\chi)\ll\frac{y}{n^{1/2}}\int_{-T_0}^{T_0}|F(it,\chi)|dt+T_0^{1/2}\int_{T_0\leqq|\lambda|\leqq T}|F(it,\chi)|\frac{dt}{T_0+|t|}+O(1).$$

Thus, the left-hand side of (3.8) is upper bounded by

$$\ll \max_{R\leqq P/2} \frac{y}{n^{1/2}} L^c \sum_{\substack{r\sim R\\ g|r}} [g,r]^{-3/2+\epsilon} \sum_{\chi}^{*} \int_{-T_0}^{T_0} |F(it,\chi)|dt$$

$$+ \max_{R\leqq P/2} \max_{0\leqq\Delta\leqq\frac{1}{RQ}} \max_{T_0\leqq T_2\leqq T} L^c T_0^{1/2} T_2^{-1} \sum_{\substack{r\sim R\\ g|r}} [g,r]^{-3/2+\epsilon} \sum_{\chi}^{*} \int_{-T_2}^{T_2} |F(it,\chi)|dt$$

$$+ \frac{R^2}{g^{3/2-\epsilon}}. \tag{3.11}$$

The term $\frac{R^2}{g^{3/2-\epsilon}}$ is permissible for $R \leqq n^{3/20-\epsilon}$ in view of (3.8). For the first two terms, we apply again Lemma 3.4:

$$\ll \max_{R\leqq P/2} \max_{\substack{d|g\\ d\leqq R}} \frac{y}{n^{1/2}} L^c g^{-3/2+\epsilon} \left(\frac{R}{d}\right)^{-3/2+\epsilon} \sum_{\substack{r\sim R\\ d|r}} \sum_{\chi}^{*} \int_{-T_0}^{T_0} |F(it,\chi)|dt$$

$$+ \max_{R\leqq P/2} \max_{T_0\leqq T_2\leqq T} \max_{\substack{d|g\\ d\leqq R}} L T_0^{1/2} T_2^{-1} g^{-3/2+\epsilon} \left(\frac{R}{d}\right)^{-3/2+\epsilon}$$

$$\times \sum_{\substack{r\sim R\\ d|r}} \sum_{\chi}^{*} \int_{-T_2}^{T_2} |F(it,\chi)|dt \ll g^{-3/2+\epsilon} y L^c,$$

for $P \leqq n^{3/20+\epsilon}$.

**3.3 Proof of Lemma 3.3.** We treat separately the cases $R \leqq L^V$ and $R \geqq L^V$ for a sufficiently large $V$ to be determined later. In the second case, we argue as in the previous section using Lemma 3.4 for $g = 1$ and use $R \geqq L^V$. In the case $R \leqq L^V$, we argue as in [7] and use the zero expansion of the von Mangoldt-function [13]:

$$\sum_{m\leqq u} \Lambda(m)\chi(m) = E_0(\chi)u - \sum_{|\operatorname{Im}\rho|\leqq T} \frac{u^\rho}{\rho} + O\left(\left(\frac{u}{T}+1\right)\log^2(uT)\right),$$

where $\rho$ runs over the non-trivial zeros of the $L$-function corresponding to $\chi \mod r$ with $|\operatorname{Im}\rho| \leqq T$. Thus,

$$W(\lambda,\chi) = \int_{N_1}^{N_2} e(u^2\lambda) d\left\{\sum_{n\leqq u} \Lambda(m)\chi(m) - E_0(\chi)u\right\}$$

$$= \int_{N_1}^{N_2} e(u^2\lambda) \sum_{|\operatorname{Im}\rho| \leqq n^{1/6}} u^{\rho-1} du + O(n^{1/3}(1+|\lambda|n)L^2)$$
$$\ll y \sum_{|\operatorname{Im}\rho| \leqq n^{1/6}} n^{(\beta-1)/2} + O(yL^{-A}),$$

where $\beta = \operatorname{Re}\rho$ and $T = n^{1/6}$. We now use the fact that $L(\sigma + it, \chi)$ with $\chi \mod r$ and $r \leqq L^V$ has no zeros in the region (see [13], VIII Satz 6.2)

$$\sigma \geqq 1 - \delta(T) := 1 - \frac{c_0}{\log r + (\log(T+2))^{4/5}}, \qquad |t| \leqq T, \tag{3.12}$$

where $c_0$ is an absolute constant. We also appeal to a well known zero density estimate [12]:

**Lemma 3.5.** *Let $N^*(\alpha, T, q)$ denote the number of zeros $\sigma + it$ of all $L$-functions to primitive characters modulo $q$ within the region $\sigma \geqq \alpha$, $|t| \leqq T$. Then*

$$\sum_{q \leqq Q} N^*(\alpha, T, q) \ll (Q^2T)^{12(1-\alpha)/5} (\log Q^2T)^c.$$

Using (3.12) and Lemma 3.5, we obtain for $T = n^{1/6}$, and such $\delta(n^{1/6}) = cL^{-4/5}$ :

$$\max_{R \leqq L^V} \sum_{r \sim R} r^{-3/2+\epsilon} \sideset{}{^*}\sum_{\chi (\operatorname{mod} r)} \max_{|\lambda| \leqq 1/rQ} |W(\lambda, \chi)|$$
$$\ll yL^V \sum_{r \leqq L^V} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|Im\,\rho| \leqq n^{1/6}} n^{\frac{\beta-1}{2}} + yL^{-A}$$
$$\ll yL^V \int_0^{1-\delta(n^{1/6})} n^{\frac{\beta-1}{2}} d\left\{ - \sum_{r \leqq L^V} N^*(\beta, n^{1/6}, r) \right\} + yL^{-A}$$
$$\ll yL^V \max_{0 \leqq \beta \leqq 1-\delta(n^{1/6})} (n^{1/6})^{\left(\frac{12}{5}+\epsilon\right)(1-\beta)} n^{\frac{(\beta-1)}{2}} + yL^{-A}$$
$$\ll yn^{-\frac{\delta(n^{1/6})}{10}}$$
$$\ll yL^{-A},$$

for any $A > 0$. □

**4. Treatment of the major arcs.** Splitting the summation over $m$ in residue classes modulo $q$ we obtain

$$S\left(\frac{a}{q} + \lambda\right) = \frac{C(a,q)}{\phi(q)} T(\lambda) + \frac{1}{\phi(q)} \sum_{\chi \bmod q} C(a,\chi) W(\lambda,\chi) + O(L^2).$$

Thus we derive from (2.2) that

$$R_1(n) = R_1^m(n) + R_1^e(n) + O(y^4 x^{-1/2} L^{-B}), \tag{4.1}$$

where

$$R_1^m(n) = \sum_{q \leqq P} \frac{1}{\phi^5(q)} \sum_{a=1}^{q*} C^5(a,q) e\left(-\frac{a}{q}n\right) \int_{-1/Qq}^{1/Qq} T^5(\lambda) e(-n\lambda) d\lambda, \tag{4.2}$$

$$\begin{aligned}
R_1^e(n) = {} & \\
& \sum_{q \leqq P} \frac{1}{\phi^5(q)} \sum_{a=1}^{q\,*} \int_{-1/Qq}^{1/Qq} \left(\sum_{\chi} C(a,\chi) W(\lambda,\chi)\right)^5 e\left(-\frac{a}{q}n - \lambda n\right) d\lambda \\
& + 5 \sum_{q \leqq P} \frac{1}{\phi^5(q)} \sum_{a=1}^{q\,*} \int_{-1/Qq}^{1/Qq} C(a,q) T(\lambda) \\
& \qquad \left(\sum_{\chi} C(a,\chi) W(\lambda,\chi)\right)^4 e\left(-\frac{a}{q}n - \lambda n\right) d\lambda \\
& + 10 \sum_{q \leqq P} \frac{1}{\phi^5(q)} \sum_{a=1}^{q\,*} \int_{-1/Qq}^{1/Qq} (C(a,q) T(\lambda))^2 \\
& \qquad \left(\sum_{\chi} C(a,\chi) W(\lambda,\chi)\right)^3 e\left(-\frac{a}{q}n - \lambda n\right) d\lambda \\
& + 10 \sum_{q \leqq P} \frac{1}{\phi^5(q)} \sum_{a=1}^{q\,*} \int_{-1/Qq}^{1/Qq} (C(a,q) T(\lambda))^3 \\
& \qquad \left(\sum_{\chi} C(a,\chi) W(\lambda,\chi)\right)^2 e\left(-\frac{a}{q}n - \lambda n\right) d\lambda \\
& + 5 \sum_{q \leqq P} \frac{1}{\phi^5(q)} \sum_{a=1}^{q\,*} \int_{-1/Qq}^{1/Qq} (C(a,q) T(\lambda))^4 \\
& \qquad \left(\sum_{\chi} C(a,\chi) W(\lambda,\chi)\right) e\left(-\frac{a}{q}n - \lambda n\right) d\lambda
\end{aligned}$$

$$=: \sum_1 + 5 \sum_2 + 10 \sum_3 + 10 \sum_4 + 5 \sum_5 . \tag{4.3}$$

We first evaluate the main term $R_1^m$. We will use the following lemmas:

**Lemma 4.1.** *Let $f(x)$, $g(x)$ be monotonic functions in the interval $[a,b]$ and $|g(x)| \ll M$. If $|f'(x)| \leqq \theta < 1$, $g(x), g'(x) \ll 1$,*

$$\sum_{a<n\leqq b} g(n)e(f(n)) = \int_a^b g(x)\, e(f(x))\, dx + O\left(\frac{1}{1-\theta}\right).$$

Proof. See [14]. □

**Lemma 4.2.** *Let $\chi_i$ mod $r_i$ with $= 1, 2, 3, 4, 5$ be primitive characters, $r = [r_1, r_2, r_3, r_4, r_5]$, and $\chi_0$ the principal character mod $q$. Then*

$$\frac{|Z(q, \chi_0\chi_1, \chi_0\chi_2, \chi_0\chi_3, \chi_0\chi_4, \chi_0\chi_5)|}{\phi^5(q)} \ll r^{-3/2+\epsilon}(\log\ P)^c.$$

Proof. See Lemma 3.3 in [2].

We apply Lemma 4.1 to $T(\lambda)$ and find

$$T(\lambda) = \int_{N_1}^{N_2} e(\lambda u^2)d\,u + O(1) = \frac{1}{2}\int_{N_1^2}^{N_2^2} v^{-1/2}e(\lambda v)d\,v + O(1)$$
$$= \frac{1}{2}\sum_{N_1^2<m\leqq N_2^2} \frac{e(\lambda m)}{\sqrt{m}} + O(1).$$

Substituting this into $R_1^m(n)$ we see

$$R_1^m(n) = \frac{1}{32}\sum_{q\leqq P}\frac{Y(q)}{\phi^5(q)}\int_{-1/Qq}^{1/Qq}\left(\sum_{N_1^2<m\leqq N_2^2}\frac{e(\lambda m)}{\sqrt{m}}\right)^5 e(-n\lambda)d\lambda$$
$$+O\left(\sum_{q\leqq P}\frac{|Y(q)|}{\phi^5(q)}\int_{-1/Qq}^{1/Qq}\left|\sum_{N_1^2<m\leqq N_2^2}\frac{e(\lambda m)}{\sqrt{m}}\right|^4 d\,\lambda\right). \tag{4.4}$$

Using

$$\sum_{N_1^2<m\leqq N_2^2}\frac{e(\lambda m)}{\sqrt{m}} \ll \min\left(y,\ \frac{1}{\sqrt{n}\|\lambda\|}\right) \tag{4.5}$$

and Lemma 4.2 with $r = 1$ we derive from (4.4)

$$\begin{aligned}
R_1^m(n) &= \frac{1}{32}\sum_{q\leqq P}\frac{Y(q)}{\phi^5(q)}\int_{-1/2}^{1/2}\left(\sum_{N_1^2<m\leqq N_2^2}\frac{e(\lambda m)}{\sqrt{m}}\right)^5 \\
&\quad e(-n\lambda)d\,\lambda + O(y^4 n^{-1/2}L^{-B}) \\
&\quad + O\left(\sum_{q\leqq P}\left|\frac{Y(q)}{\phi^5(q)}\right|\int_{1/Qq}^{1/2}\frac{1}{(\sqrt{n}|\lambda|)^5}d\lambda\right) \\
&= \frac{1}{32}P_0\sum_{q\leqq P}\frac{Y(q)}{\phi^5(q)} + O(P^5Q^4n^{-5/2}) + O(y^4n^{-1/2}L^{-B}) \\
&= \frac{1}{32}P_0\sum_{q\leqq P}\frac{Y(q)}{\phi^5(q)} + O(y^4n^{-1/2}L^{-B}). \qquad (4.6)
\end{aligned}$$

for any $B > 0$, where $P_0$ is defined as in (2.6). Applying Lemma 4.2, we can estimate $\sum_1$ in the following way:

$$\begin{aligned}
\left|\sum_1\right| &= \Bigg|\sum_{q\leqq P}\frac{1}{\phi^5(q)}\sum_{\chi_1 \bmod q}\sum_{\chi_2 \bmod q}\sum_{\chi_3 \bmod q}\sum_{\chi_4 \bmod q}\sum_{\chi_5 \bmod q} \\
&\quad Z(q,\chi_1,\chi_2,\chi_3,\chi_4,\chi_5)\int_{-1/Qq}^{1/Qq}\prod_{j=1}^{5}W(\lambda,\chi_j)e(-n\lambda)d\lambda\Bigg| \\
&\leqq \sum_{r_1\leqq P}\sum_{r_2\leqq P}\sum_{r_3\leqq P}\sum_{r_4\leqq P}\sum_{\substack{r_5\leqq P\\ [r_1,r_2,r_3,r_4,r_5]\leqq P}}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\sideset{}{^*}\sum_{\chi_2 \bmod r_2} \\
&\quad \sideset{}{^*}\sum_{\chi_3 \bmod r_3}\sideset{}{^*}\sum_{\chi_4 \bmod r_4}\sideset{}{^*}\sum_{\chi_5 \bmod r_5}\int_{-1/Q[r_1,r_2,r_3,r_4,r_5]}^{1/Q[r_1,r_2,r_3,r_4,r_5]}\prod_{j=1}^{5}\left|W(\lambda,\chi_j)\right|d\,\lambda \\
&\quad \times \sum_{\substack{q\leqq P\\ [r_1,r_2,r_3,r_4,r_5]|q}}\frac{|Z(q,\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0,\chi_4\chi_0,\chi_5\chi_0)|}{\phi^5(q)} \\
&\ll L^c\sum_{r_1\leqq P}\sum_{r_2\leqq P}\sum_{r_3\leqq P}\sum_{r_4\leqq P}\sum_{r_5\leqq P}[r_1,r_2,r_3,r_4,r_5]^{-\frac{3}{2}+\epsilon} \\
&\quad \sideset{}{^*}\sum_{\chi_1 \bmod r_1}\sideset{}{^*}\sum_{\chi_2 \bmod r_2}\sideset{}{^*}\sum_{\chi_3 \bmod r_3}
\end{aligned}$$

$$\sideset{}{^*}\sum_{\chi_4 \bmod r_4} \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \int\limits_{-1/Q[r_1,r_2,r_3,r_4,r_5]}^{1/Q[r_1,r_2,r_3,r_4,r_5]} \prod_{i=1}^{5} |W(\lambda, \chi_i)| d\lambda$$

$$\leqq L^c \sum_{r_1 \leqq P} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \max_{|\lambda| \leqq 1/r_1 Q} |W(\lambda, \chi_1)| \sum_{r_2 \leqq P} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \max_{|\lambda| \leqq 1/r_2 Q} |W(\lambda, \chi_2)|$$

$$\times \sum_{r_3 \leqq P} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \max_{|\lambda| \leqq 1/r_3 Q} |W(\lambda, \chi_3)| \sum_{r_4 \leqq P} \sideset{}{^*}\sum_{\chi_4 \bmod r_4} \left( \int\limits_{-1/Qr_4}^{1/Qr_4} |W(\lambda, \chi_4)|^2 d\lambda \right)^{1/2}$$

$$\times \sum_{r_5 \leqq P} \sideset{}{^*}\sum_{\chi_5 \bmod r_5} [r_1, r_2, r_3, r_4, r_5]^{-\frac{3}{2}+\epsilon} \left( \int\limits_{-1/Qr_5}^{1/Qr_5} |W(\lambda, \chi_5)|^2 d\lambda \right)^{1/2}. \tag{4.7}$$

Since $s = [r_1, r_2, r_3, r_4, r_5] = [[r_1, r_2, r_4, r_5], r_3]$, we use Lemma 3.1 to estimate the sum over $r_5$ in (4.7) is

$$= \sum_{r_5 \leqq P} [[r_1, r_2, r_3, r_4], r_5]^{-3/2+\epsilon} \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \left( \int\limits_{-1/r_5 Q}^{1/r_5 Q} |W(\lambda, \chi_5)|^2 d\lambda \right)^{1/2}$$
$$\leqq K([r_1, r_2, r_3, r_4]) \ll [r_1, r_2, r_3, r_4]^{-3/2+\epsilon} y^{1/2} n^{-1/4} L^c.$$

Applying again Lemma 3.1, we see that the contribution of this quantity to the sum over $r_4$ is:

$$\ll y^{1/2} n^{-1/4} L^c \sum_{r_4 \leqq P} [r_1, r_2, r_3, r_4]^{-3/2+\epsilon} \sideset{}{^*}\sum_{\chi_4 \bmod r_4} \left( \int\limits_{-1/r_4 Q}^{1/r_4 Q} |W(\lambda, \chi_4)|^2 d\lambda \right)^{1/2}$$
$$= K([r_1, r_2, r_3]) y^{1/2} n^{-1/4} L^c \ll [r_1, r_2, r_3]^{-3/2+\epsilon} y n^{-1/2} L^c.$$

Applying now Lemma 3.2, we see that the contribution of this quantity to the sum over $r_3$ is:

$$\ll y n^{-1/2} L^c \sum_{r_3 \leqq P} [r_1, r_2, r_3]^{-3/2+\epsilon} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \max_{|\lambda| \leqq 1/r_3 Q} |W(\lambda, \chi_3)|$$
$$\leqq J([r_1, r_2]) y n^{-1/2} L^c \ll [r_1, r_2]^{-3/2+\epsilon} y^2 n^{-1/2} L^c.$$

Applying Lemma 3.2 again, we see

$$\ll y^2 n^{-1/2} L^c \sum_{r_2 \leqq P} [r_1, r_2]^{-3/2+\epsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \max_{|\lambda| \leqq 1/r_2 Q} |W(\lambda, \chi_2)|$$
$$\leqq y^2 n^{-1/2} L^c J([r_1]) \ll [r_1]^{-3/2+\epsilon} y^3 n^{-1/2} N L^c.$$

Inserting the last bound into (4.8) and using Lemma 3.3, we estimate the sum over $r_1$ as follows:

$$\begin{aligned}\sum_1 &\ll y^3 n^{-1/2} L^c \sum_{r_1 \leqq P} r_1^{-3/2+\epsilon} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \max_{|\lambda| \leqq 1/r_1 Q} |W(\lambda, \chi_1)| \\ &= y^3 n^{-1/2} L^c J(1) \ll y^4 n^{-1/2} L^{-A}. \qquad (4.8)\end{aligned}$$

For the estimation of the sums $\sum_2$–$\sum_5$ we note that

$$T = \max_{|\lambda| \leqq 1/Q} |T(\lambda)| \ll y,$$

and using (4.5) we get:

$$\left( \int_{-1/Q}^{1/Q} |T(\lambda)|^2 \, d\lambda \right)^{1/2} \ll y^{1/2} n^{-1/4}.$$

Using these estimates and the Lemmas 3.1–3.3, we argue similarly to the estimation of $\sum_1$ and obtain:

$$\sum_2 + \sum_3 + \sum_4 + \sum_5 \ll y^4 n^{-1/2} L^{-A}. \qquad (4.9)$$

Thus, we see from (4.1), (4.3), (4.6), (4.7), and (4.8):

$$R_1^m(n) = \frac{1}{32} P_0 \sum_{q \geqq 1} A(q) + O(y^4 n^{-1/2} L^{-B}),$$

i.e., we proved (2.5). □

**5. Proof of Theorem 1.** We now derive (2.7) from (2.5). We use

**Lemma 5.1.**

$$\sum_{q \leqq P} A(q) = \prod_{p \leqq P} s(p) + O(P^{-1/2+\epsilon}),$$

*where* $\prod_{p \leqq P} s(p) > c > 0$.

Proof. This is Lemma 4.2 in [8]. Applying Lemma 5.1 to (2.5) yields (2.7). □

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